Computational Methods
CMSC/AMSC/MAPL 460

Eigenvalues and Eigenvectors

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Eigen Values of a Matrix

- Already met eigenvalues and eigenvectors a few times in the course
- Here we will study them more formally
- Definition:
- A $N \times N$ matrix **A** has an eigenvector **x** (non-zero) with corresponding eigenvalue λ if

$$\mathbf{A}\mathbf{x}=\lambda\,\mathbf{x}$$

- This means

$$\mathbf{A}\mathbf{x}-\lambda\,\mathbf{x}=0 \qquad\qquad (\mathbf{A}-\lambda I)\,\mathbf{x}=0$$

- If two numbers multiply to zero one of them is zero
- If a matrix vector product gives a zero vector, then either the vector is zero, or the matrix has zero determinant (is singular).

Solving for eigenvalues

- The zero vector is not an eigenvector (nothing special about $\mathbf{A}0=0$)
- So we need $(\mathbf{A}-\lambda \mathrm{I})\,\mathbf{x}=0 \qquad \|\mathbf{x}\|_2 \neq 0$

$$\det(\mathbf{A}-\lambda\,\mathbf{I})=0$$

- Evaluating the determinant we get an Nth degree polynomial equation, which can be solved for N roots
  - Could be solved numerically using zero finding algorithms
- So a $N \times N$ matrix has N eigenvalues
- Of course eigenvalues need not be distinct.
- E.g. eigenvalues of identity matrix are given by solution of

$$(1-\lambda)^n=0$$

- So the matrix has N repeated eigenvalues equal to 1

Assorted properties of eigenvalues & eigenvectors

- Shift eigenvalues of a matrix by τ.
  - Let

$$\mathbf{Ax}=\lambda\ \mathbf{x}$$

  - Add $-\tau\ \mathbf{x}$ to both sides

$$(\mathbf{A}-\tau\ \mathbf{I})\mathbf{x}=(\lambda-\tau)\ \mathbf{x}$$

  - We get a new matrix

$$\mathbf{B}=(\mathbf{A}-\tau\ \mathbf{I})$$

  - Shifted eigenvalue $(\lambda-\tau)$
  - Same eigenvector $\mathbf{x}$
- Eigenvectors are not normalized:
  - If $\mathbf{x}$ is an eigenvector so is $\alpha\ \mathbf{x}$.
  - Often eigenvectors are normalized to have $\|\mathbf{x}\|_2=1$
- The term *eigenvalue* is a partial translation of the German "eigenvert." A complete translation would be something like "own value" or "characteristic value" .

Eigenvalues and eigenvector

- Recall a $N \times N$ matrix maps N dimensional vectors to other N dimensional vectors
  - In general it maps elements in R^N to other elements in R^N
- Eigenvectors and eigenvalues provide basic information about this mapping
  - Identify special vectors which remain untransformed (or just scaled)
- Important in many areas
  - Quantum mechanics – energy levels
  - Acoustics – fundamental frequencies of drums or columns
  - Stability theory – resonant frequencies or critical values of parameters

Eigen-value decomposition

- Represent the matrix in terms of its eigenvalues and eigenvectors
- A N$\times$ N matrix has N eigenvalues and eigen vectors
- Write the *N* equations

$$\mathbf{A}\mathbf{x}_i = \lambda_i \mathbf{x}_i$$

- by stacking the vectors $\mathbf{x}_i$ as columns of a matrix **X** and the constants λ_i along the diagonal of a matrix
- We get

$$\mathbf{AX} = \mathbf{X}\Lambda$$

- If all eigenvectors are independent, then $\mathbf{X}^{-1}$ exists, and so

$$\mathbf{X}^{-1}\mathbf{AX} = \mathbf{X}^{-1}\mathbf{X}\Lambda = \Lambda$$

$$\mathbf{A} = \mathbf{X}\Lambda\mathbf{X}^{-1}$$

- This is the eigenvalue decomposition of a matrix **A**

Use of the eigenvalue decomposition

- Can use it to study the properties of A
- Recall condition number definition

$$\operatorname{cond}(A) = \kappa(A) = \|A\| \cdot \|A^{-1}\|$$

$$= \frac{\text{maximum stretch}}{\text{maximum shrink}} = \frac{\max_i |\lambda_i|}{\min_i |\lambda_i|}$$

- Natural frequencies of the matrix
- Powers of a matrix

$$\mathbf{Ax} = \lambda\ \mathbf{x}$$

$$\mathbf{A}(\mathbf{Ax}) = \mathbf{A}\ (\lambda\ \mathbf{x}) = \lambda\ \mathbf{Ax} = \lambda^2\ \mathbf{x}$$

$$\mathbf{A}^n\ \mathbf{x} = \lambda^n\ \mathbf{x}$$

- Apply same idea to EVD

$$\mathbf{A}^n = \mathbf{X}\Lambda^n\mathbf{X}^{-1}$$

Similar Matrices

- Two matrices **A** and **B** are said to be similar it is possible to relate them as

$$\mathbf{B}=\mathbf{T}^{-1}\mathbf{A}\mathbf{T} \qquad\qquad \mathbf{T}\mathbf{B}\mathbf{T}^{-1}=\mathbf{A}$$

- Here **T** is any non singular matrix, which is the similarity transform matrix
- Theorem: Similar matrices have the same eigenvalues and eigen-vectors that are related via the similarity transform.
- Proof. Let $(\mathbf{x},\lambda)$ be an eigen-pair for **A**. Then $\mathbf{A}\mathbf{x}=\lambda\,\mathbf{x}$.

Let $y=T^{-1}x$ and $x=Ty$

Then $Ax=\lambda\,\mathbf{x}$.

Premultiply by T^{-1} to get $T^{-1}ATT^{-1}\,x = \lambda\, T^{-1}x$

So $\mathbf{B}\mathbf{y}=\lambda\,\mathbf{y}$

EVD

- EVD is a similarity transform that takes A to a diagonal matrix using a matrix of eigenvectors.
- Eigenvalue decomposition requires solving of a general polynomial equation.
  - Even if matrix has real entries eigenvalues can be complex
  - So can eigenvectors
- Eigenvectors provide a set of basis vectors in which the matrix becomes diagonal

Example from the book

- Let A = [-149 -50 -154; 537 180 546; -27 -9 -25]
- This matrix was constructed in such a way that the characteristic polynomial factors nicely.
- $\det(A-\lambda I) = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda - 1)(\lambda - 2)(\lambda - 3)$
- Consequently the three eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$, and
- Λ = [1 0 0; 0 2 0; 0 0 3]
- The matrix of eigenvectors can be normalized so that its elements are all integers.
- X =[1 -4 7 ; - 3 9 – 49; 0 1 9 1]
- It turns out that the inverse of X also has integer entries.
- X^{-1} = [130 43 133 ; 27 9 28 ; -3 -1 -3]
- These matrices provide the eigenvalue decomposition of our example.
- $A = X\Lambda X^{-1}$

Eigshow

- Eigen values of 2×2 matrix represent transformations in the plane
- Ideas of symmetry

Left and Right Eigenvectors

- So far we just talked about matrix products

$$Ax=\lambda\ x$$

- For a $N\times N$ matrix we can also define a left matrix product

$$y^t\ A=v$$

- So if we have

$$y^t\ A=\lambda\ y$$

then y is a left eigenvector of A

- If A is symmetric $A=A^t$
- $(Ax)^t = x^t A^t = x^t A$
- So left and right eigenvectors of a symmetric matrix are the same

Symmetric Matrices

- A matrix is symmetric if its transpose is equal to itself
- **A** is symmetric if $\mathbf{A}^t = \mathbf{A}$
- Eigenvalues and Eigenvectors of a real symmetric matrix are real.
- Eigenvectors are orthogonal.
- $\mathbf{A} \cdot \mathbf{X}_R = \mathbf{X}_R \cdot \mathrm{diag}(\lambda 1 \ldots \lambda N)$
- $\mathbf{X}_\mathrm{L} \cdot \mathbf{A} = \mathrm{diag}(\lambda 1 \ldots \lambda N) \cdot \mathbf{X}_\mathrm{L}$
- Multiply first equation on left by $\mathbf{X}_\mathrm{L}$,
- second on the right by $\mathbf{X}_\mathrm{R}$, and subtract
- $(\mathbf{X}_L \cdot \mathbf{X}_\mathrm{R}) \cdot \mathrm{diag}(\lambda 1 \ldots \lambda N) = \mathrm{diag}(\lambda 1 \ldots \lambda N) \cdot (\mathbf{X}_\mathrm{L} \cdot \mathbf{X}_\mathrm{R})$
- matrix of dot products of the left and right eigenvectors commutes with the diagonal matrix of eigenvalues.
- Only matrices that commute with a diagonal matrix *of distinct elements* are themselves diagonal.