Computational Methods
CMSC/AMSC/MAPL 460

Ordinary differential equations

Ramani Duraiswami,
Dept. of Computer Science

Several slides adapted from Profs. Dianne O'Leary and Eric Sandt, TAMU

Ordinary differential equations

- Mathematical modeling involves posing models and then solving these models numerically or analytically
- ODEs represent a powerful method of modeling
  - Especially if things depend on rates of change
- Rate of change of distance is velocity

$$v = dx/dt \qquad x(t) = \int_0^t \frac{dx}{d\tau} d\tau$$

- Knowing the velocity as a function of τ we can integrate using numerical quadrature
- Rate of change of velocity is acceleration

$$a = dv/dt = d^2x/dt^2$$

- Given initial conditions ($v(0)=0$, $x(0)=0$) find the location $x(t)$ at time t given that the object falls with a constant acceleration of 10 m/s^2

Solution by simple integration

$$\int dv = \int 10dt$$

$$v = 10\,t + c_1$$

$$\int dx/dt\; dt = \int 10t\; dt$$

$$x = 10t^2/2 + c_1 t + c_2$$

- Use initial conditions
- $v(0)=0$ so c_1 is zero
- $x(0)=0$ so c_2 is zero
- Final solution $x=5t^2$
- Could handle more complex functions of t under the integral $\frac{dx}{dt} = f(t)$ $\qquad x(t) = \int_0^t f(\tau)d\tau$

What if simple integration would not work?

- Example: Let the velocity be a function of x and t
- *dx/dt=f(x,t)*

$$x(t) = \int_0^t f(x(\tau), \tau) d\tau$$

- Cannot be simply integrated
- This is the typical type of problem we need to solve in ODEs
- This is nonlinear because solution x depends on itself
- The linear case could be solved using numerical quadrature
  - Previous chapter

Writing a 2nd order system in standard form

- ODEs in standard form: way they are input to software
- Written as a set of 1st order equations with initial conditions

$$u'' = g(t, u, u')$$

$$u(0) = u_0$$

$$u'(0) = v_0$$

- where u_0 and v_0 are given.
- Let $y_1 = u$ and $y_2 = u'$. Then, in standard form:

$$y_2' = g(t, y_1, y_2)$$

$$y_1' = y_2$$

$$y_1(0) = u_0 \quad y_2(0) = v_0$$

Standard form

- We'll work with problems in **standard form**,

$$y' = f(t;\, y)$$

$$y(0) = y_0$$

- where the function y has m components,
- y' means the derivative with respect to t, and
- y_0 is a given vector of initial conditions (numbers).
- Writing this component-by-component yields

$$y'_{(1)} = f_1(t,\, y_{(1)} \ldots\, y_{(m)})$$

$$\ldots$$

$$y'_{(m)} = f_m(t,\, y_{(1)} \ldots\, y_{(m)})$$

with $y_{(1)}(t_0), \ldots, y_{(m)}(t_0)$ given initial conditions

A modeling exercise: predator prey problems

- Eco-system (island) that contains rabbits and foxes
- Island has plenty of food for rabbits
- Rabbits reproduce like crazy and would fill-up the island
- Foxes eat rabbits
- Let $r(t)$ represent the number of rabbits and $f(t)$ the number of foxes.
- Model the number of rabbits and foxes on the island and decide if it will reach an equilibrium

Rabbit and fox population

- Rabbit population will grow at a certain rate
  - a is the natural growth rate of rabbits in the absence of predation,
- Rabbits will die as they are eaten by foxes. Let the rabbit die if it encounters a fox.
  - b is the death rate per encounter of rabbits due to predation,
- Fox population dies off if they cannot eat rabbits
  - c is the natural death rate of foxes in the absence of food (rabbits),
- Foxes reproduce if they have food
  - e is the efficiency of turning predated rabbits into foxes.
- Initial conditions $R(0)=r_0$ *and* $F(0)=f_0$
- Volterra equations

$$dR/dt = aR - bRF$$
$$dF/dt = ebRF - cF$$

Standard form

- Volterra's model

$$dR/dt = 2R - \alpha RF$$
$$dF/dt = \alpha RF - F$$

- Another example from the book

$$\ddot{u}(t) = -u(t)/r(t)^3$$
$$\ddot{v}(t) = -v(t)/r(t)^3$$

where

$$r(t) = \sqrt{u(t)^2 + v(t)^2}$$

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function ydot = twobody(t,y)
r = sqrt(y(1)^2 + y(2)^2);
ydot = [y(3); y(4); -y(1)/r^3; -y(2)/r^3];
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The vector $y(t)$ has four components,

$$y(t) = \begin{bmatrix} u(t) \\ v(t) \\ \dot{u}(t) \\ \dot{v}(t) \end{bmatrix}$$

The differential equation is

$$\dot{y}(t) = \begin{bmatrix} \dot{u}(t) \\ \dot{v}(t) \\ -u(t)/r(t)^3 \\ -v(t)/r(t)^3 \end{bmatrix}$$

Solving differential equations: Euler's method

- As in quadrature: use Taylor series
- Euler's method

$$y(t + h) = y(t) + hy'(t) + h^2/2 \qquad y''(\xi)$$

for some point ξ *in* $[t, \cdot\ t + h]$

- Note that

$$y'(t) = f(t, y(t)):$$

- March forward

$$y(h) = y_1 = y(0) + hy'(0)$$

$$= y_0 + hf(t_0, y_0)$$

$$y_{n+1} = y_n + hf_n$$

Example

Consider

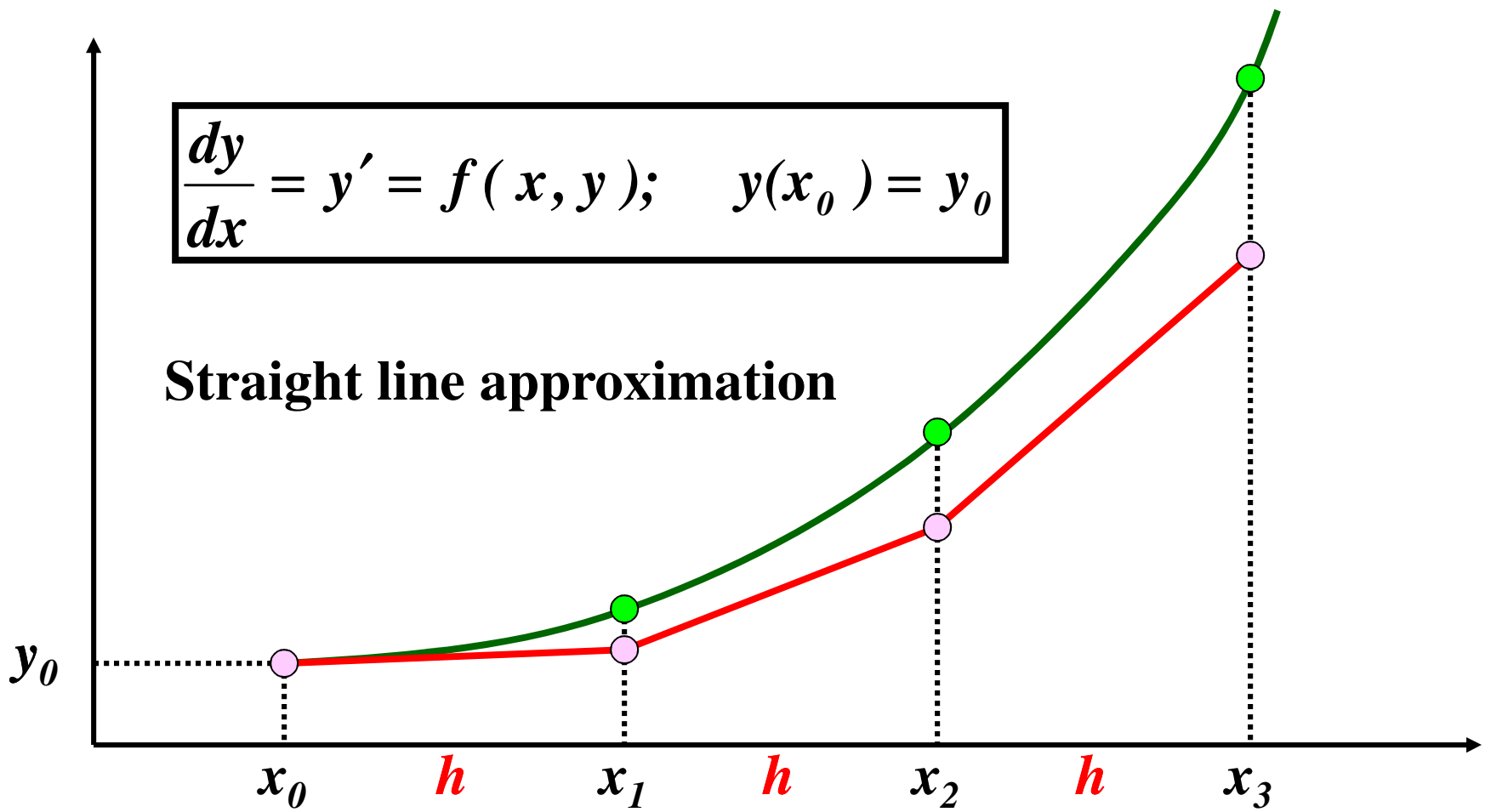
$$\frac{dy}{dx} = x + y$$

The initial conditions is :

$$y(0) = 1$$

The analytical solution $y(x) = 2e^{x} - x - 1$

Euler's Method: First-order Taylor Method



Euler's Method Example

Consider

$$\frac{dy}{dx} = x + y$$

The initial condition is: $y(0) = 1$

The step size is: $\Delta h = 0.02$

The analytical solution is:

$$y(x) = 2e^{x} - x - 1$$

Euler's Method Example

The algorithm has a loop using the initial conditions and definition of the derivative

The derivative is calculated as:

$$y_{\mathrm{i}}' = x_{\mathrm{i}} + y_{\mathrm{i}}$$

The next y value is calculated:

$$y_{\mathrm{i+1}} = y_{\mathrm{i}} + \Delta h \; y_{\mathrm{i}}'$$

Take the next step:

$$x_{\mathrm{i+1}} = x_{\mathrm{i}} + \Delta h$$

Euler's Method Example

The results

x_n	y_n	y'_n	hy'_n	Exact Solution	Error
0	1.00000	1.00000	0.02000	1.00000	0.00000
0.02	1.02000	1.04000	0.02080	1.02040	-0.00040
0.04	1.04080	1.08080	0.02162	1.04162	-0.00082
0.06	1.06242	1.12242	0.02245	1.06367	-0.00126
0.08	1.08486	1.16486	0.02330	1.08657	-0.00171
0.1	1.10816	1.20816	0.02416	1.11034	-0.00218
0.12	1.13232	1.25232	0.02505	1.13499	-0.00267
0.14	1.15737	1.29737	0.02595	1.16055	-0.00318
0.16	1.18332	1.34332	0.02687	1.18702	-0.00370
0.18	1.21019	1.39019	0.02780	1.21443	-0.00425
0.2	1.23799	1.43799	0.02876	1.24281	-0.00482

Euler's Method

The trouble with this method is

- Lack of accuracy
- Small step size needed for accuracy