Error analysis

- The formulas all have the form

$$Q(f) = \sum_{i=1}^{m} \alpha_i f(t_i)$$

- The error function

$$R(f) = I(f) - Q(f)$$

is a **linear operator**; i.e., for every two functions f and g, and for every two scalars β and γ,

$$R(\beta f + \gamma g) = \beta R(f) + \gamma R(g)\,.$$

(We restrict f and g to lie in some function space; for example, we need a certain number of continuous derivatives in order for the polynomial error formula to apply.)

Formula for trapezoidal rule

If f(t) and its 1st two derivatives are continuous on [a,b], then

$$\int_a^b f(t)dt - T = -\frac{(b-a)^3}{12} f''(\eta)$$

where $\eta \in [a, b]$.

Proof: The trapezoidal rule is computed by integrating the linear interpolant to $f(t)$ at a and b.

From our work on polynomial interpolation, we know that, for the linear interpolant,

$$f(t) - p(t) = f[a, b, t](t-a)(t-b) \, ,$$

so

$$\int_a^b f(t)dt - T \quad = \quad \int_a^b f[a, b, t](t-a)(t-b)dt$$

Error trapezoidal rule

Recall the **Integral Mean Value Theorem**: If $w(t)$ doesn't change sign on $[a,b]$ then

$$\int_a^b w(t)f(t) = f(\xi)\int_a^b w(t)dt$$

for some point $\xi \in [a,b]$.

Therefore,

$$\begin{aligned}\int_a^b f(t)dt - T &= f[a,b,\xi]\int_a^b (t-a)(t-b)dt \\ &= f[a,b,\xi](-\frac{1}{6}(b-a)^3)\end{aligned}$$

The result follows from the fact that

$$f[a,b,\xi] = \frac{1}{2}f''(\eta)\;. \square$$

How to reduce error?

- If (b-a) is large error is higher
  - Use composite rules.
  - As we saw they give lower error

Example: Composite Trapezoidal Rule. Let's divide $[a, b]$ into n pieces of equal length $h = (b-a)/n$.

$$\int_a^b f(t)dt$$

$$\approx \frac{h}{2}(f(a)+f(a+h))+\frac{h}{2}(f(a+h)+f(a+2h)+\ldots+\frac{h}{2}(f(a+(n-1)h)+f(a+nh))$$

$$= h[\frac{1}{2}f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)h) + \frac{1}{2}f(b)]$$

$$\equiv T_n\,.$$

Error of composite trapezoid

The **Error formula** for the Composite Trapezoidal Rule is

$$\int_a^b f(t)dt - T_n = -\sum_{i=1}^{n} \frac{h^3}{12} f''(\eta_i)$$

where $\eta_i \in [a + (i-1)h, a + ih]$. Since $nh = b - a$, and

$$\frac{1}{n}\sum_{i=1}^{n} f''(\eta_i)$$

is an average value of f'' on $[a, b]$, we obtain

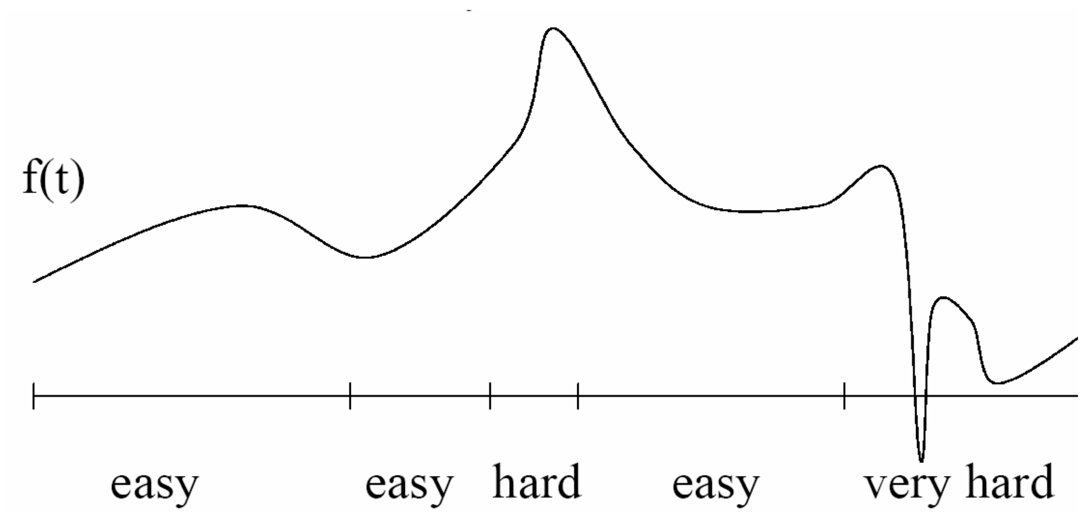
$$\int_a^b f(t)dt - T_n = -\frac{(b-a)h^2}{12} f''(\eta)$$

for some $\eta \in [a, b]$. □

- Error decreases by a factor of $(b\text{-}a)^2$

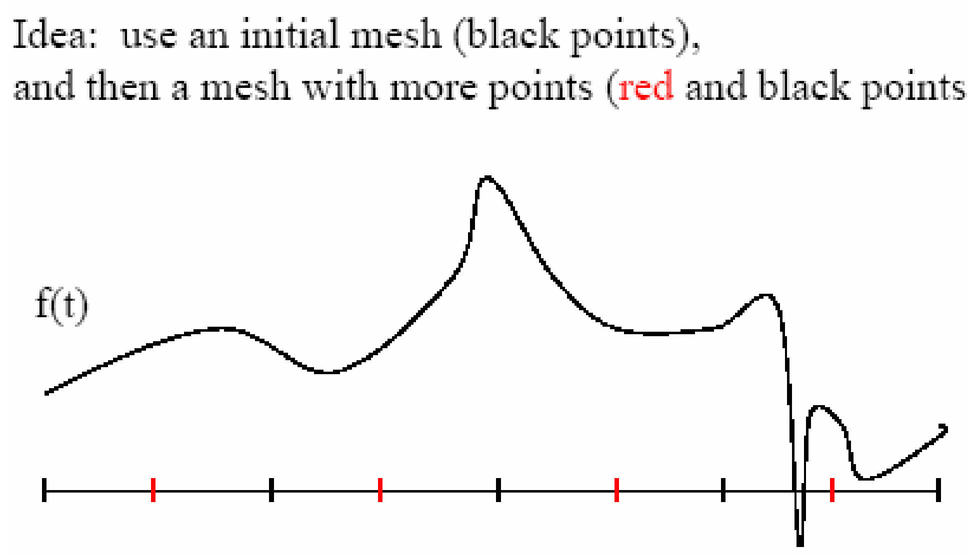
Adaptive integration

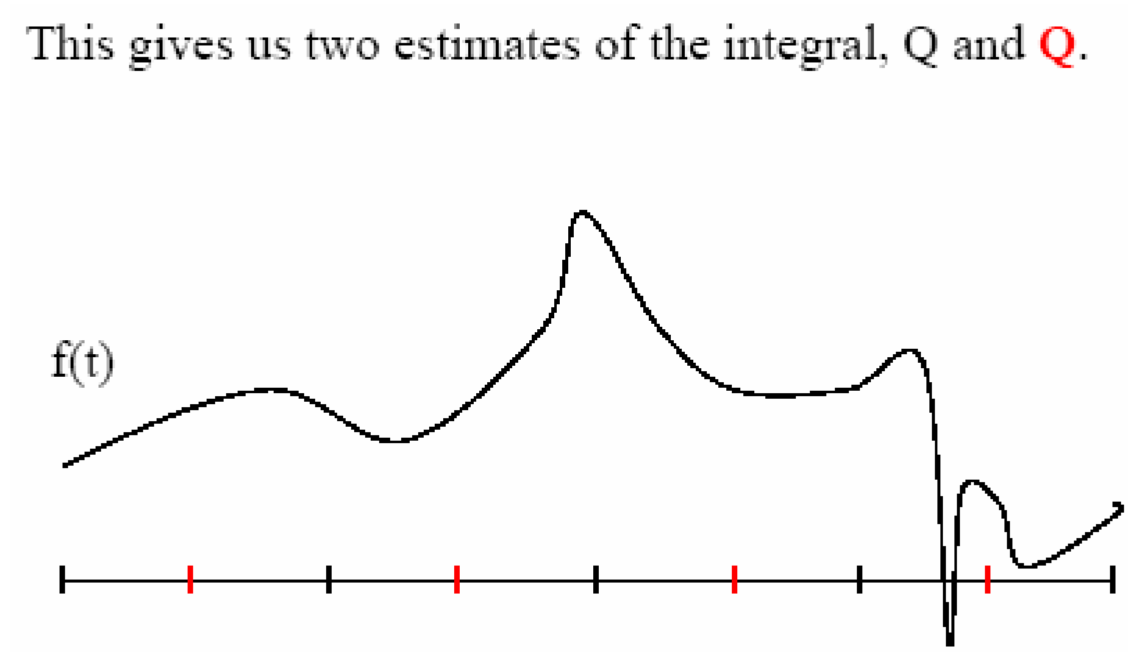
- Other factor in the error is the second derivative
- Idea, keeping the error fixed, reduce the size of the interval where second derivative is high
- If we used the worst part of the domain to determine step size we would waste resources on the easy parts
- Idea of adaptive use different *h* for different parts



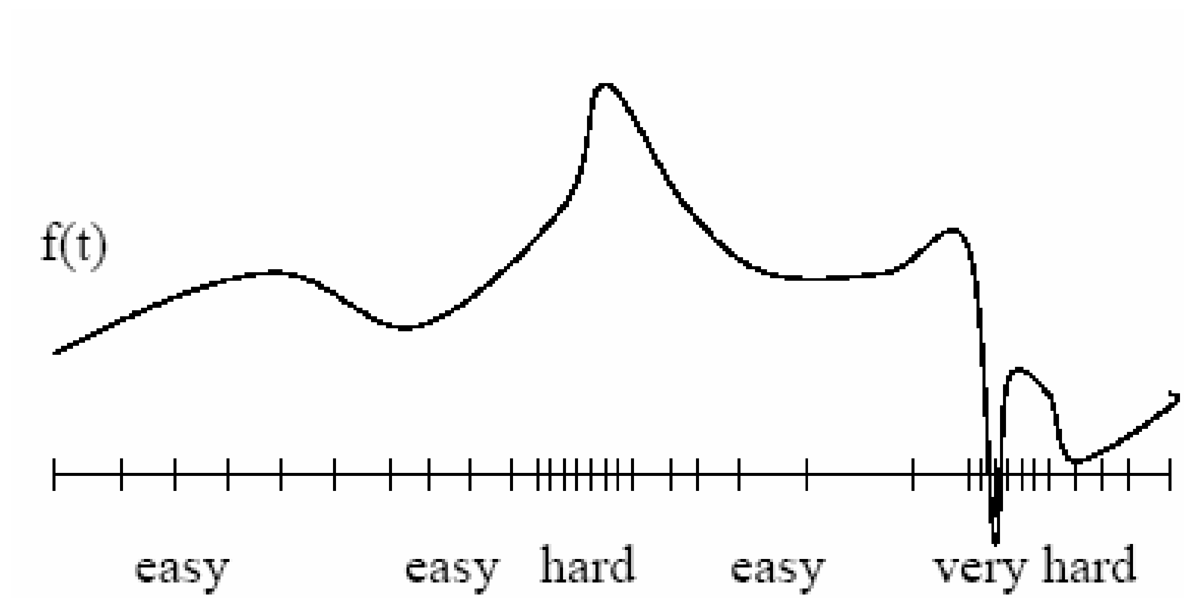
Adaptive integration

- In general we do not have a graph to tell us where things are bad.
- Need a function which estimates the error locally
- Idea: use two formulae: one more accurate, and one less accurate in each interval and estimate the error
- Difference gives an estimate of the error locally
- Where error is larger we need to do something





- If local error estimate is less than tolerance in a particular region we can stop dividing it.
- Otherwise split the interval in two pieces, and repeat the procedure
- Each sub-interval tolerance requirement needs to be half that of the parents
- Upon convergence each subinterval achieves success.
  - Some subintervals needed lots of points, others few
- Add up all sub interval answers and report to calling program



Adaptive methods

- Allow us to achieve a given tolerance at a given cost