Error analysis

- The formulas all have the form

\[ Q(f) = \sum_{i=1}^{m} \alpha_i f(t_i) \]

- The error function

\[ R(f) = I(f) - Q(f) \]

is a linear operator; i.e., for every two functions \( f \) and \( g \), and for every two scalars \( \beta \) and \( \gamma \),

\[ R(\beta f + \gamma g) = \beta R(f) + \gamma R(g) . \]

(We restrict \( f \) and \( g \) to lie in some function space; for example, we need a certain number of continuous derivatives in order for the polynomial error formula to apply.)
Formula for trapezoidal rule

If \( f(t) \) and its 1\(^{\text{st}} \) two derivatives are continuous on \([a,b]\), then

\[
\int_a^b f(t)dt - T = -\frac{(b-a)^3}{12} f''(\eta)
\]

where \( \eta \in [a,b] \).

**Proof:** The trapezoidal rule is computed by integrating the linear interpolant to \( f(t) \) at \( a \) and \( b \).

From our work on polynomial interpolation, we know that, for the linear interpolant,

\[
f(t) - p(t) = f[a, b, t](t - a)(t - b),
\]

so

\[
\int_a^b f(t)dt - T = \int_a^b f[a, b, t](t - a)(t - b)dt
\]
Error trapezoidal rule

Recall the **Integral Mean Value Theorem**: If $w(t)$ doesn’t change sign on $[a, b]$ then

$$\int_a^b w(t)f(t) = f(\xi) \int_a^b w(t)dt$$

for some point $\xi \in [a, b]$.

Therefore,

$$\int_a^b f(t)dt - T = f[a, b, \xi] \int_a^b (t-a)(t-b)dt$$

$$= f[a, b, \xi] \left(-\frac{1}{6}(b-a)^3\right)$$

The result follows from the fact that

$$f[a, b, \xi] = \frac{1}{2} f''(\eta).$$
How to reduce error?

- If \((b-a)\) is large error is higher
  - Use composite rules.
  - As we saw they give lower error

**Example:** Composite Trapezoidal Rule. Let’s divide \([a, b]\) into \(n\) pieces of equal length \(h = (b - a)/n\).

\[
\int_{a}^{b} f(t) dt \\
\approx \frac{h}{2}(f(a)+f(a+h)) + \frac{h}{2}(f(a+h)+f(a+2h)) + \ldots + \frac{h}{2}(f(a+(n-1)h)+f(a+nh))
\]

\[
= h\left[\frac{1}{2}f(a) + f(a + h) + f(a + 2h) + \ldots + f(a + (n - 1)h) + \frac{1}{2}f(b)\right]
\]

\[\equiv T_n.\]
Error of composite trapezoid

The **Error formula** for the Composite Trapezoidal Rule is

\[
\int_a^b f(t)dt - T_n = -\sum_{i=1}^{n} \frac{h^3}{12} f''(\eta_i)
\]

where \( \eta_i \in [a + (i - 1)h, a + ih] \). Since \( nh = b - a \), and

\[
\frac{1}{n} \sum_{i=1}^{n} f''(\eta_i)
\]

is an average value of \( f'' \) on \([a, b] \), we obtain

\[
\int_a^b f(t)dt - T_n = -\frac{(b-a)h^2}{12} f''(\eta)
\]

for some \( \eta \in [a, b] \). 

- Error decreases by a factor of \((b-a)^2\)
Adaptive integration

- Other factor in the error is the second derivative
- Idea, keeping the error fixed, reduce the size of the interval where second derivative is high
- If we used the worst part of the domain to determine step size we would waste resources on the easy parts
- Idea of adaptive use different $h$ for different parts
Adaptive integration

- In general we do not have a graph to tell us where things are bad.
- Need a function which estimates the error locally.
- Idea: use two formulae: one more accurate, and one less accurate in each interval and estimate the error.
- Difference gives an estimate of the error locally.
- Where error is larger we need to do something.
• If local error estimate is less than tolerance in a particular region we can stop dividing it.
• Otherwise split the interval in two pieces, and repeat the procedure
• Each sub-interval tolerance requirement needs to be half that of the parents
• Upon convergence each subinterval achieves success.
  – Some subintervals needed lots of points, others few
• Add up all sub interval answers and report to calling program
Adaptive methods

- Allow us to achieve a given tolerance at a given cost