

*Computational Methods*  
CMSC/AMSC/MAPL 460

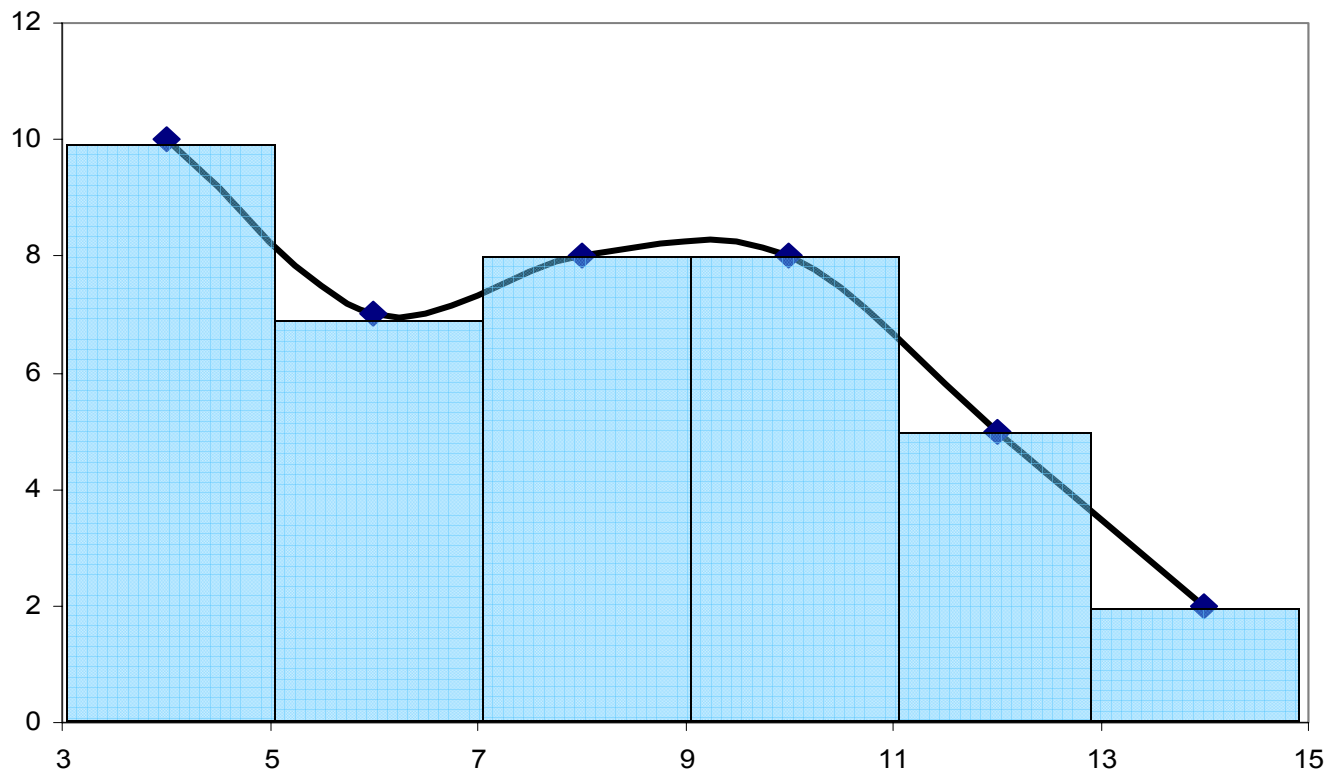
Quadrature: Integration

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# *Numerical Integration*

Idea is to do integral in small parts, like the way you first learned integration - **a summation**

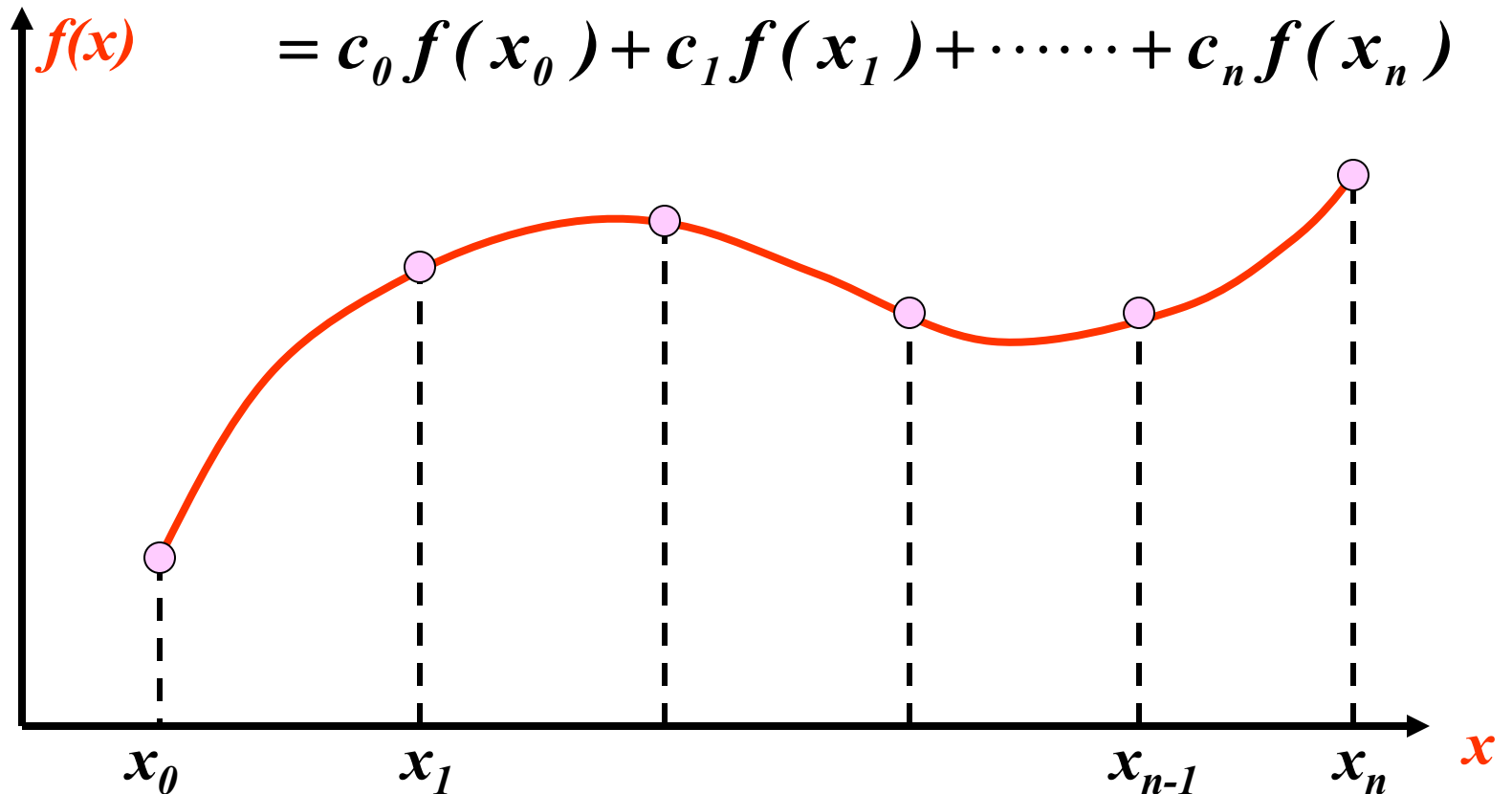
Numerical methods just try to make it faster and more accurate



# *Basic Numerical Integration*

- **Weighted sum of function values**

$$\int_a^b f(x) dx \approx \sum_{i=0}^n c_i f(x_i)$$



# ***Numerical Integration***

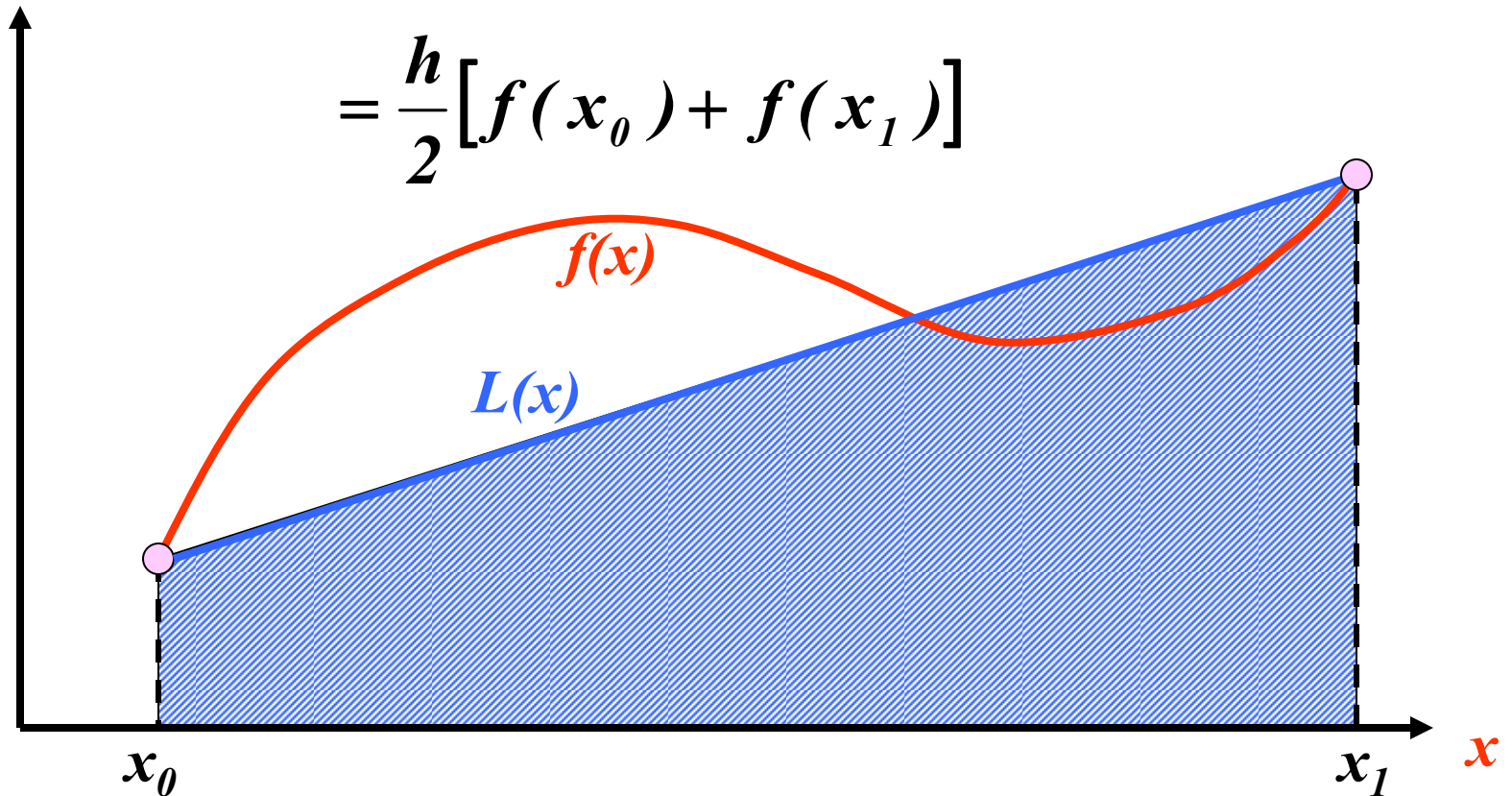
- **Characterized by where the function is evaluated**
- **Newton-Cotes Closed Formulae -- Use both end points**
  - Trapezoidal Rule : Linear
  - Simpson's 1/3-Rule : Quadratic
  - Simpson's 3/8-Rule : Cubic
  - Boole's Rule : Fourth-order
- **Newton-Cotes Open Formulae -- Use only interior points**
  - midpoint rule

# Trapezoid Rule

- Straight-line approximation

$$\int_a^b f(x) dx \approx \sum_{i=0}^1 c_i f(x_i) = c_0 f(x_0) + c_1 f(x_1)$$

$$= \frac{h}{2} [f(x_0) + f(x_1)]$$



# *Example: Trapezoid Rule*

Evaluate the integral  $\int_0^4 xe^{2x} dx$

- **Exact solution (integration by parts)**

$$\begin{aligned}\int_0^4 xe^{2x} dx &= \left[ \frac{x}{2} e^{2x} - \frac{1}{4} e^{2x} \right]_0^4 \\ &= \frac{1}{4} e^{2x} (2x - 1) \Big|_0^4 = 5216.926477\end{aligned}$$

- **Trapezoidal Rule**

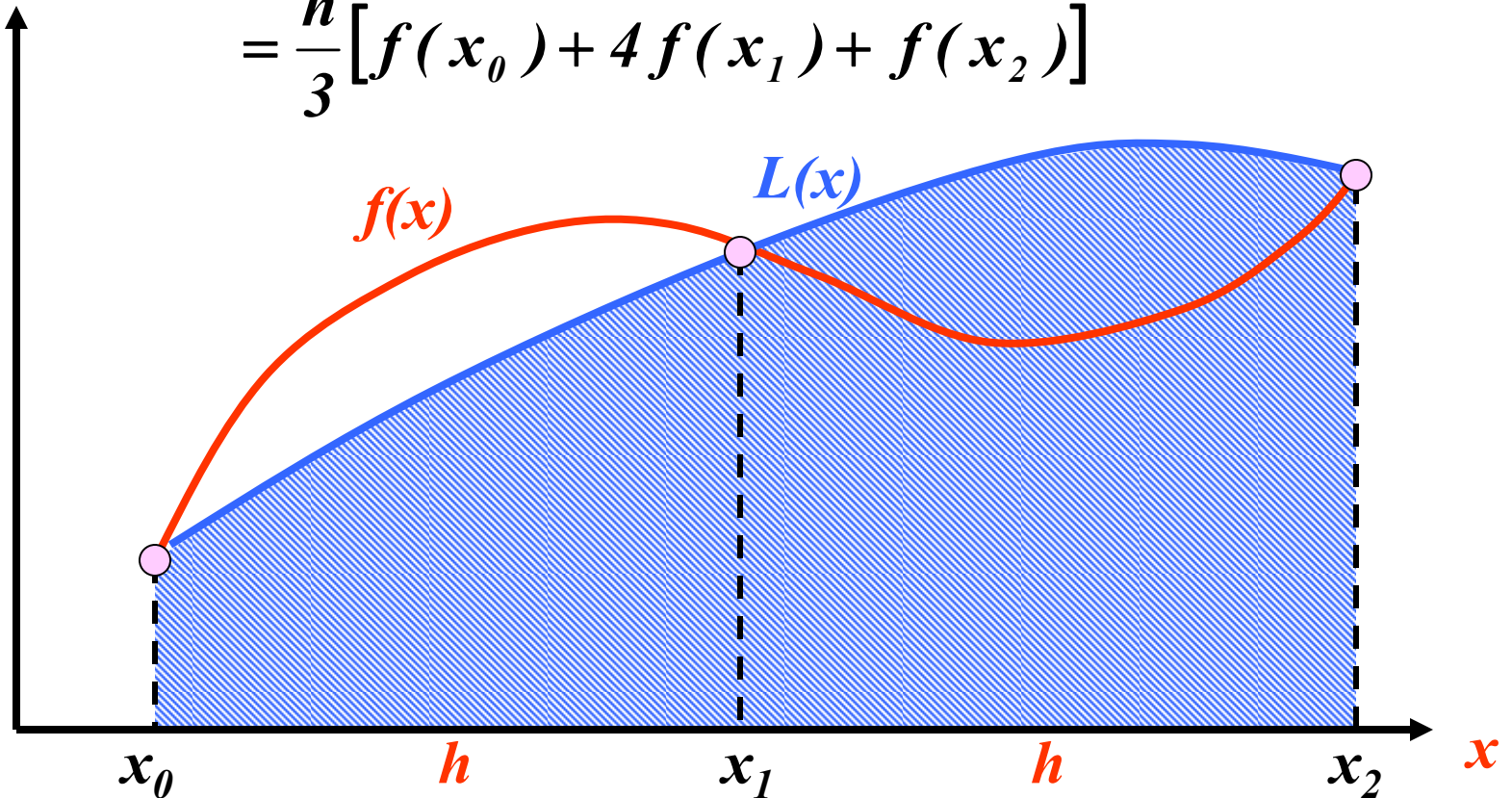
$$I = \int_0^4 xe^{2x} dx \approx \frac{4-0}{2} [f(0) + f(4)] = 2(0 + 4e^8) = 23847.66$$

$$\varepsilon = \frac{5216.926 - 23847.66}{5216.926} = -357.12\%$$

# *Simpson's 1/3-Rule*

Approximate the function by a parabola

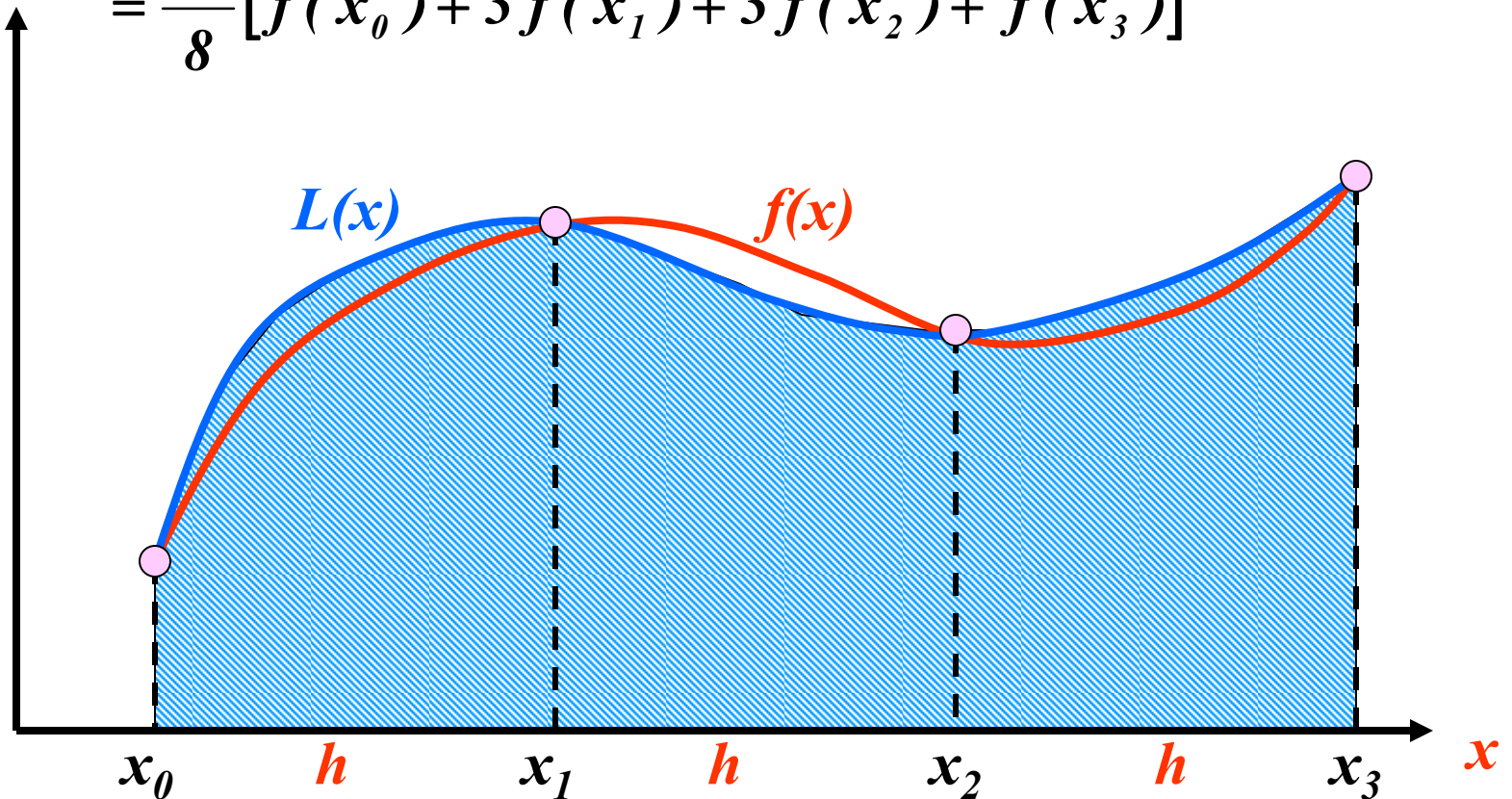
$$\int_a^b f(x) dx \approx \sum_{i=0}^2 c_i f(x_i) = c_0 f(x_0) + c_1 f(x_1) + c_2 f(x_2)$$
$$= \frac{h}{3} [f(x_0) + 4f(x_1) + f(x_2)]$$



# *Simpson's 3/8-Rule*

Approximate by a cubic polynomial

$$\int_a^b f(x) dx \approx \sum_{i=0}^3 c_i f(x_i) = c_0 f(x_0) + c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3)$$
$$= \frac{3h}{8} [f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)]$$



# *Example: Simpson's Rules*

Evaluate the integral

$$\int_0^4 xe^{2x} dx$$

- **Simpson's 1/3-Rule**

$$\begin{aligned} I &= \int_0^4 xe^{2x} dx \approx \frac{h}{3} [f(0) + 4f(2) + f(4)] \\ &= \frac{2}{3} [0 + 4(2e^4) + 4e^8] = 8240.411 \\ \varepsilon &= \frac{5216.926 - 8240.411}{5216.926} = -57.96\% \end{aligned}$$

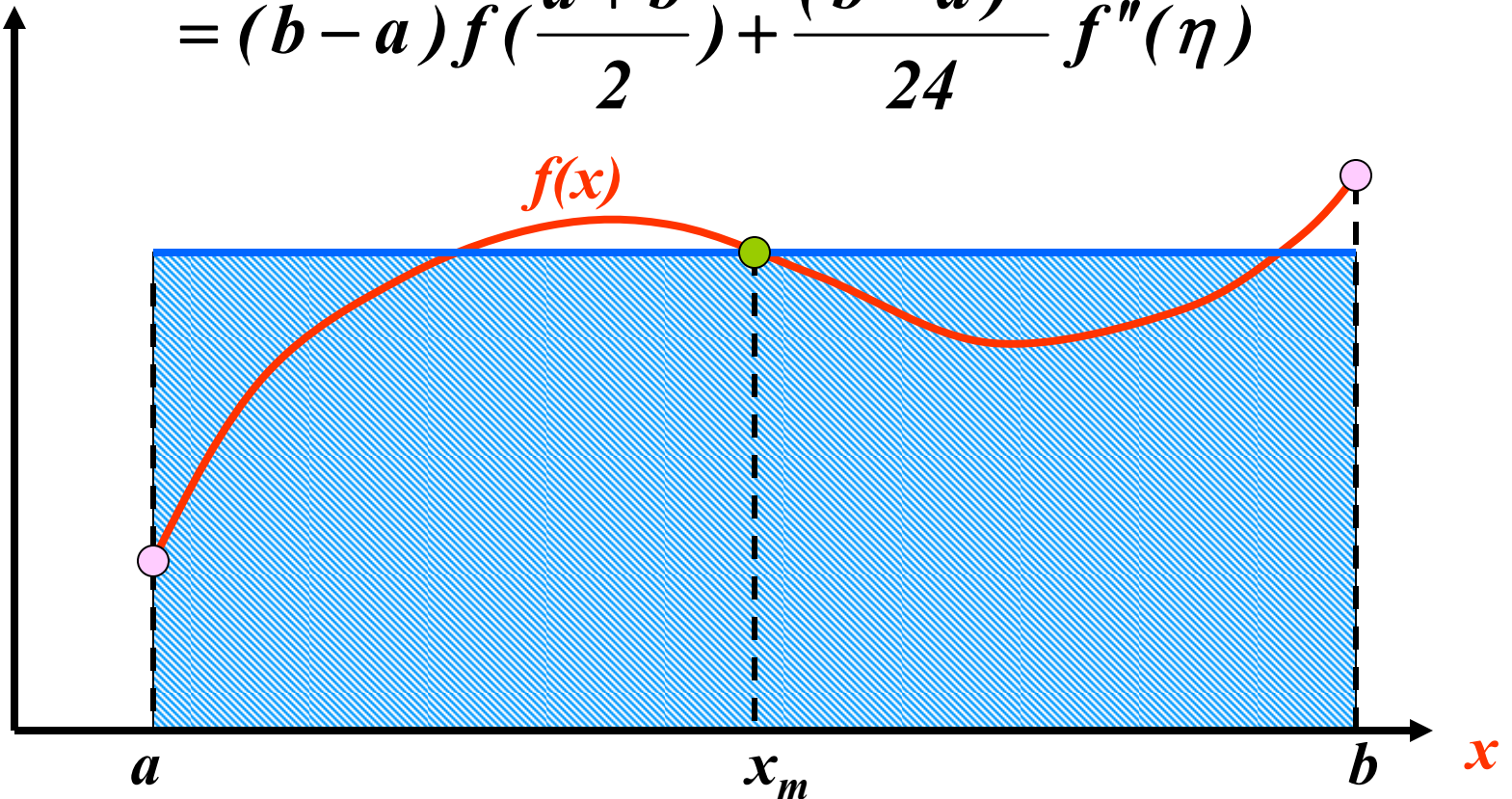
- **Simpson's 3/8-Rule**

$$\begin{aligned} I &= \int_0^4 xe^{2x} dx \approx \frac{3h}{8} \left[ f(0) + 3f\left(\frac{4}{3}\right) + 3f\left(\frac{8}{3}\right) + f(4) \right] \\ &= \frac{3(4/3)}{8} [0 + 3(19.18922) + 3(552.33933) + 11923.832] = 6819.209 \\ \varepsilon &= \frac{5216.926 - 6819.209}{5216.926} = -30.71\% \end{aligned}$$

# *Midpoint Rule*

## Newton-Cotes Open Formula

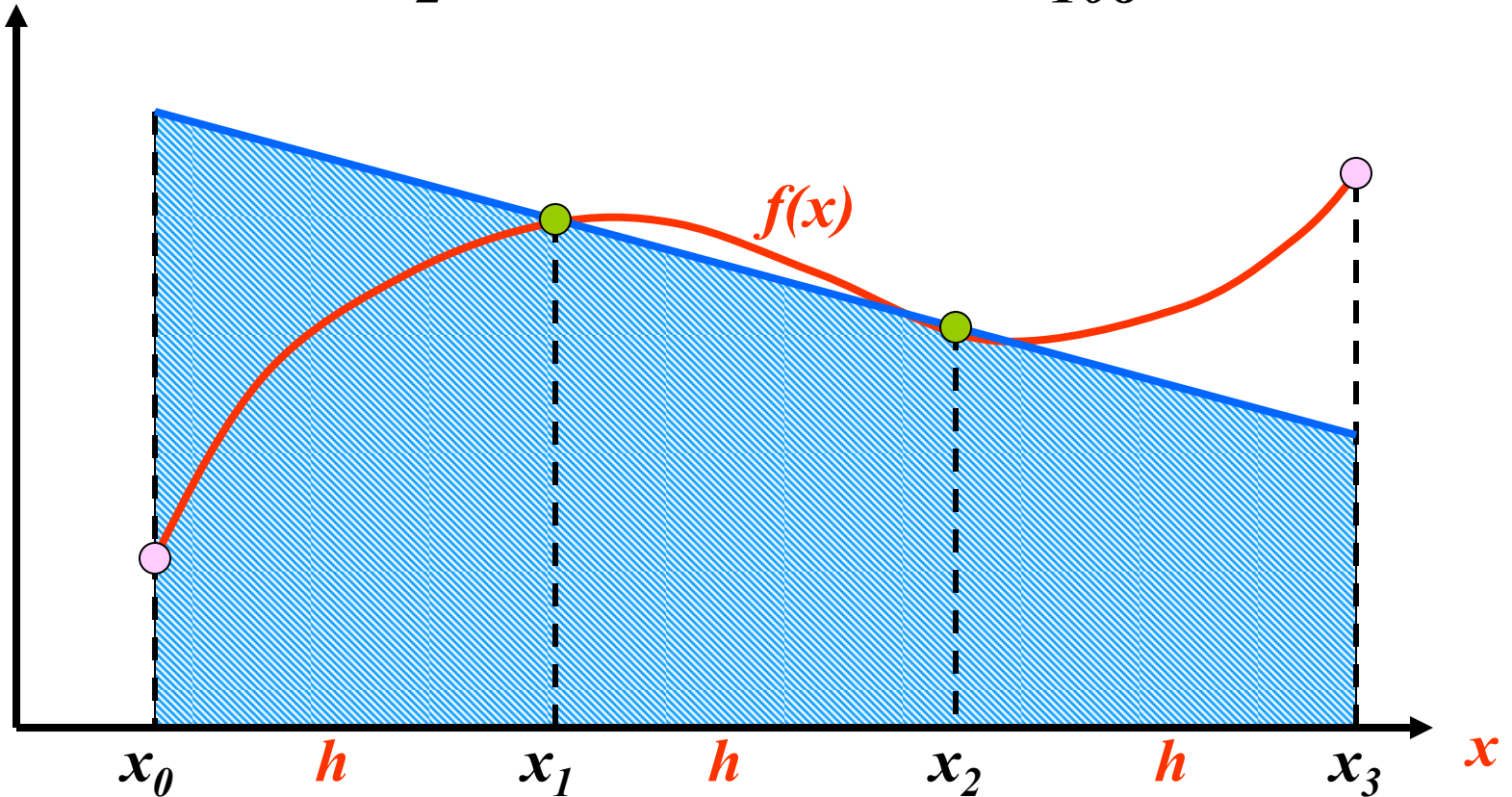
$$\int_a^b f(x) dx \approx (b-a) f(x_m)$$
$$= (b-a) f\left(\frac{a+b}{2}\right) + \frac{(b-a)^3}{24} f''(\eta)$$



# *Two-point Newton-Cotes Open Formula*

Approximate by a straight line

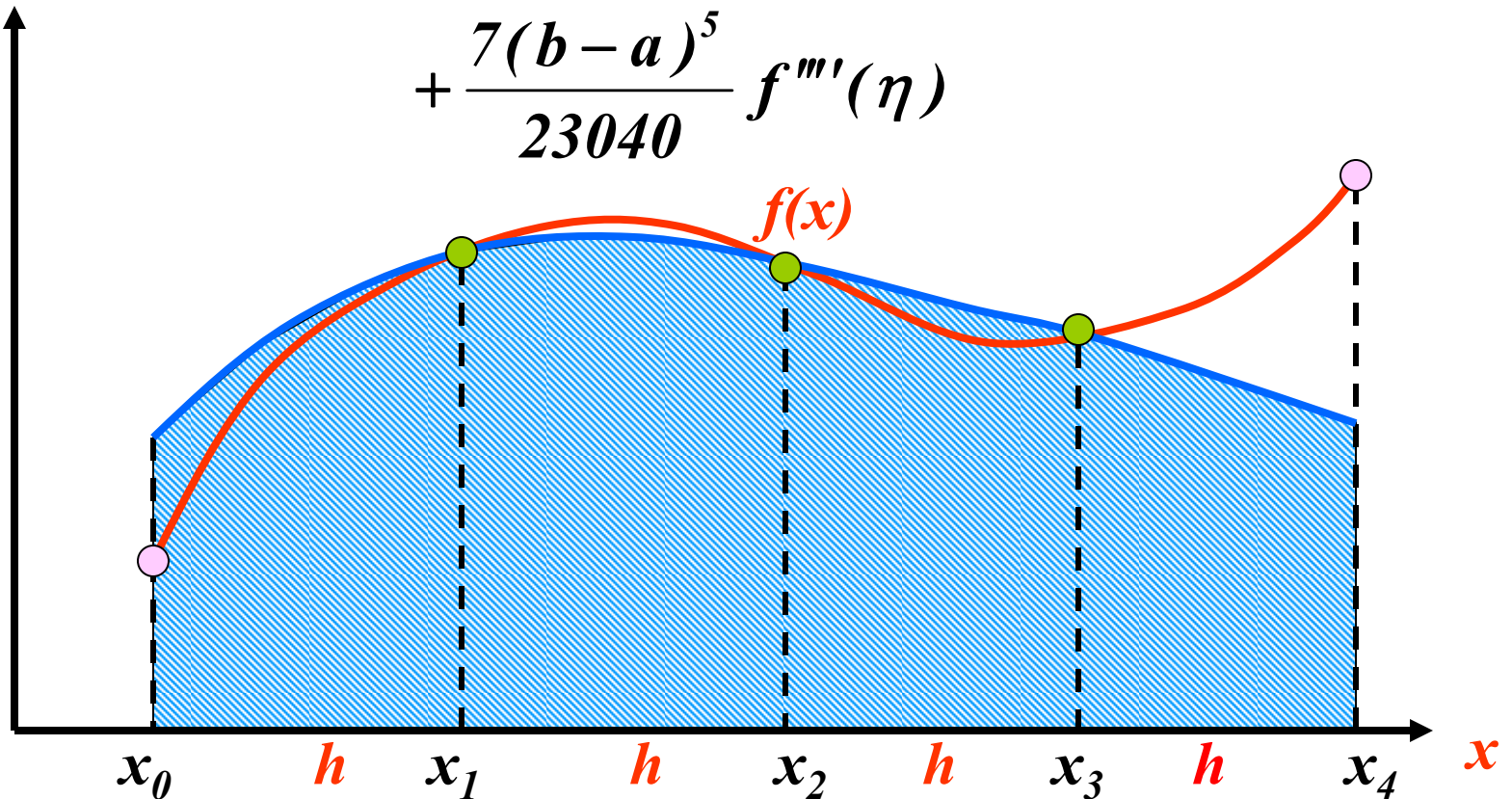
$$\int_a^b f(x) dx \approx \frac{b-a}{2} [f(x_1) + f(x_2)] + \frac{(b-a)^3}{108} f''(\eta)$$



# *Three-point Newton-Cotes Open Formula*

Approximate by a parabola

$$\int_a^b f(x) dx \approx \frac{b-a}{3} [2f(x_1) - f(x_2) + 2f(x_3)] + \frac{7(b-a)^5}{23040} f'''(\eta)$$



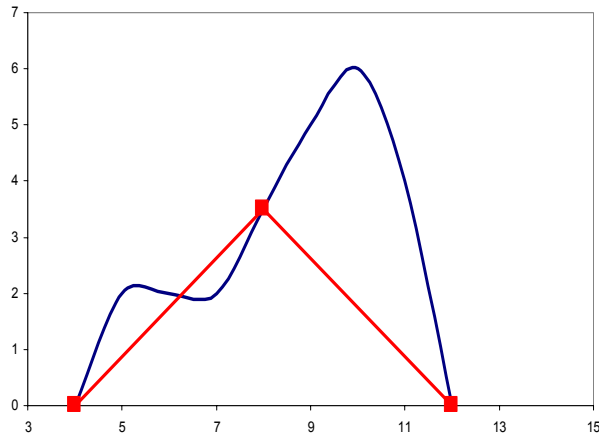
- $F_1 * (x^3/3 - 5h * x^2/2 + 6 * h^2 * x) / (2h^2)$
- $F_1 * ((4h)^3/3 - 5h * (4h)^2/2 + 6h^2 * 4 * h) / (2h^2)$
- $F_1 * (4h)(8/3 - 5 + 3)$
- $F_1 * (4h/3)(8 + 9 - 15)$
- $(b-a)/3 * F_1 * 2$

# ***Better Numerical Integration***

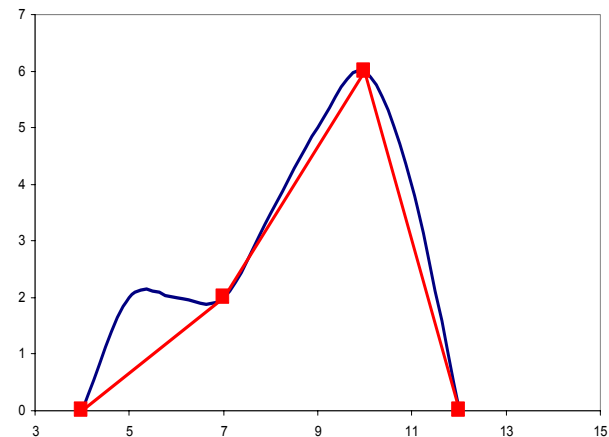
- **Composite integration**
  - **Composite Trapezoidal Rule**
  - **Composite Simpson's Rule**
- **Richardson Extrapolation**
- **Romberg integration**

# Apply trapezoid rule to multiple segments over integration limits

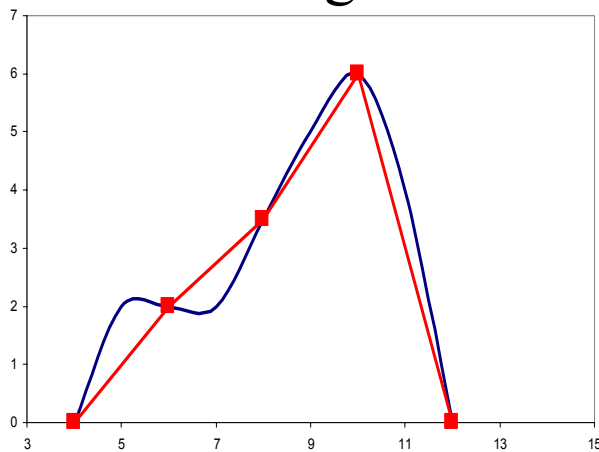
Two segments



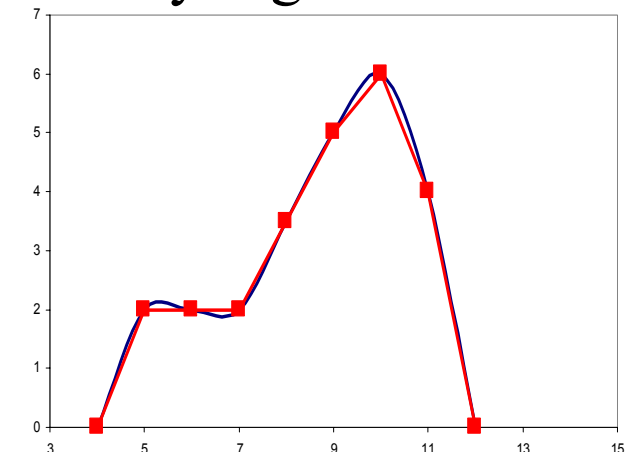
Three segments



Four segments

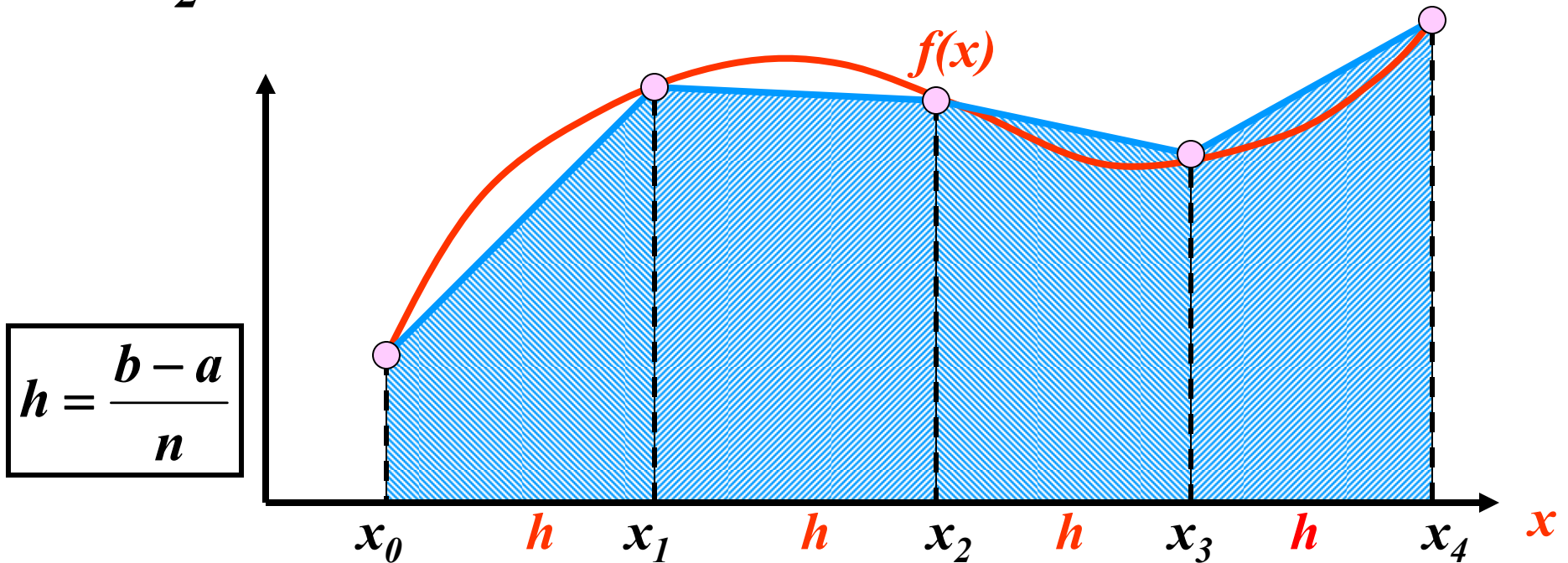


Many segments



# Composite Trapezoid Rule

$$\begin{aligned}\int_a^b f(x)dx &= \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)dx \\ &= \frac{h}{2}[f(x_0) + f(x_1)] + \frac{h}{2}[f(x_1) + f(x_2)] + \cdots + \frac{h}{2}[f(x_{n-1}) + f(x_n)] \\ &= \frac{h}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_i) + \cdots + 2f(x_{n-1}) + f(x_n)]\end{aligned}$$



# *Composite Trapezoid Rule*

Evaluate the integral

$$I = \int_0^4 x e^{2x} dx$$

$$n = 1, h = 4 \Rightarrow I = \frac{h}{2} [f(0) + f(4)] = 23847.66 \quad \varepsilon = -357.12\%$$

$$n = 2, h = 2 \Rightarrow I = \frac{h}{2} [f(0) + 2f(2) + f(4)] = 12142.23 \quad \varepsilon = -132.75\%$$

$$n = 4, h = 1 \Rightarrow I = \frac{h}{2} [f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)] = 7288.79 \quad \varepsilon = -39.71\%$$

$$n = 8, h = 0.5 \Rightarrow I = \frac{h}{2} [f(0) + 2f(0.5) + 2f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + 2f(3) + 2f(3.5) + f(4)] = 5764.76 \quad \varepsilon = -10.50\%$$

$$n = 16, h = 0.25 \Rightarrow I = \frac{h}{2} [f(0) + 2f(0.25) + 2f(0.5) + \dots + 2f(3.5) + 2f(3.75) + f(4)] = 5355.95 \quad \varepsilon = -2.66\%$$

# *Composite Trapezoid Rule with Unequal Segments*

Evaluate the integral

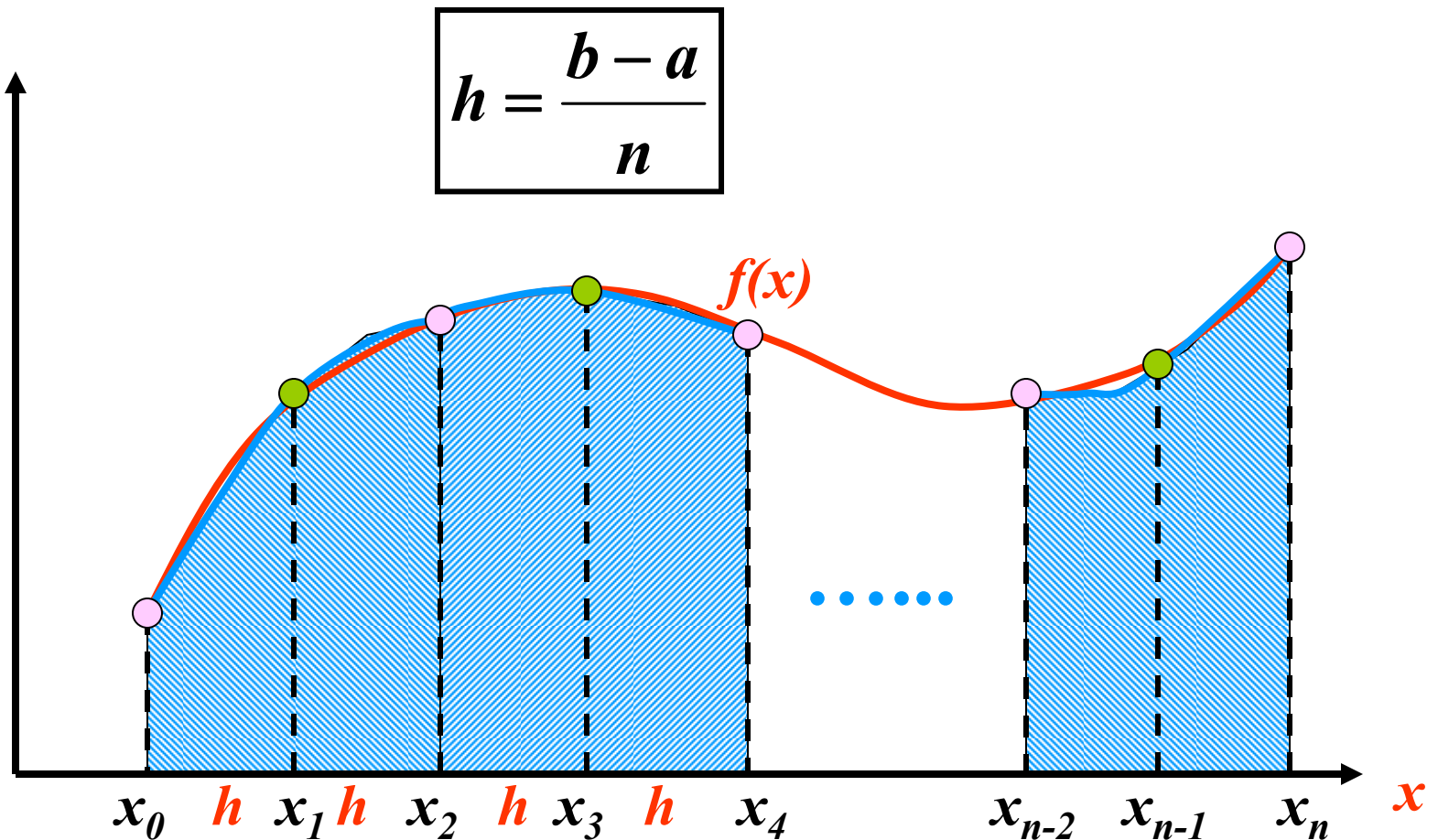
$$I = \int_0^4 xe^{2x} dx$$

- $h_1 = 2, h_2 = 1, h_3 = 0.5, h_4 = 0.5$

$$\begin{aligned} I &= \int_0^2 f(x)dx + \int_2^3 f(x)dx + \int_3^{3.5} f(x)dx + \int_{3.5}^4 f(x)dx \\ &= \frac{h_1}{2} [f(0) + f(2)] + \frac{h_2}{2} [f(2) + f(3)] \\ &\quad + \frac{h_3}{2} [f(3) + f(3.5)] + \frac{h_4}{2} [f(3.5) + f(4)] \\ &= \frac{2}{2} [0 + 2e^4] + \frac{1}{2} [2e^4 + 3e^6] + \frac{0.5}{2} [3e^6 + 3.5e^7] \\ &\quad + \frac{0.5}{2} [3.5e^7 + 4e^8] = 5971.58 \quad \Rightarrow \varepsilon = -14.45\% \end{aligned}$$

# *Composite Simpson's Rule*

## Piecewise Quadratic approximations



# *Composite Simpson's Rule*

## **Multiple applications of Simpson's rule**

$$\begin{aligned}\int_a^b f(x) dx &= \int_{x_0}^{x_2} f(x) dx + \int_{x_2}^{x_4} f(x) dx + \cdots + \int_{x_{n-2}}^{x_n} f(x) dx \\ &= \frac{h}{3} [f(x_0) + 4f(x_1) + f(x_2)] + \frac{h}{3} [f(x_2) + 4f(x_3) + f(x_4)] \\ &\quad + \cdots + \frac{h}{3} [f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]\end{aligned}$$

$$\begin{aligned}&= \frac{h}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots \\ &\quad + 4f(x_{2i-1}) + 2f(x_{2i}) + 4f(x_{2i+1}) + \cdots \\ &\quad + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]\end{aligned}$$

# *Composite Simpson's Rule*

Evaluate the integral

$$I = \int_0^4 xe^{2x} dx$$

- $n = 2, h = 2$

$$\begin{aligned} I &= \frac{h}{3} [f(0) + 4f(2) + f(4)] \\ &= \frac{2}{3} [0 + 4(2e^4) + 4e^8] = 8240.411 \Rightarrow \varepsilon = -57.96\% \end{aligned}$$

- $n = 4, h = 1$

$$\begin{aligned} I &= \frac{h}{3} [f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)] \\ &= \frac{1}{3} [0 + 4(e^2) + 2(2e^4) + 4(3e^6) + 4e^8] \\ &= 5670.975 \Rightarrow \varepsilon = -8.70\% \end{aligned}$$

# *Composite Simpson's Rule with Unequal Segments*

Evaluate the integral

$$I = \int_0^4 xe^{2x} dx$$

- $h_1 = 1.5, h_2 = 0.5$

$$\begin{aligned} I &= \int_0^3 f(x) dx + \int_3^4 f(x) dx \\ &= \frac{h_1}{3} [f(0) + 4f(1.5) + 2f(3)] \\ &\quad + \frac{h_2}{3} [f(3) + 4f(3.5) + 2f(4)] \\ &= \frac{1.5}{3} [0 + 4(1.5e^3) + 3e^6] + \frac{0.5}{3} [3e^6 + 4(3.5e^7) + 4e^8] \\ &= 5413.23 \quad \Rightarrow \varepsilon = -3.76\% \end{aligned}$$