Image Motion
The Information from Image Motion

• 3D motion between observer and scene + structure of the scene
  – Wallach O’Connell (1953): Kinetic depth effect
  – http://www.biols.susx.ac.uk/home/George_Mather/Motion/KDE.HTML
  – Motion parallax: two static points close by in the image with different
    image motion; the larger translational motion corresponds to the point
    closer by (smaller depth)

• Recognition
  – Johansson (1975): Light bulbs on joints
  – http://www.biols.susx.ac.uk/home/George_Mather/Motion/index.html
Examples of Motion Fields I

(a) Motion field of a pilot looking straight ahead while approaching a fixed point on a landing strip. (b) Pilot is looking to the right in level flight.
Examples of Motion Fields II

(a) Translation perpendicular to a surface. (b) Rotation about axis perpendicular to image plane. (c) Translation parallel to a surface at a constant distance. (d) Translation parallel to an obstacle in front of a more distant background.
Optical flow
Assuming that illumination does not change:

- Image changes are due to the RELATIVE MOTION between the scene and the camera.
- There are 3 possibilities:
  - Camera still, moving scene
  - Moving camera, still scene
  - Moving camera, moving scene
Motion Analysis Problems

- **Correspondence Problem**
  - Track corresponding elements across frames

- **Reconstruction Problem**
  - Given a number of corresponding elements, and camera parameters, what can we say about the 3D motion and structure of the observed scene?

- **Segmentation Problem**
  - What are the regions of the image plane which correspond to *different* moving objects?
Motion Field (MF)

- The **MF** assigns a velocity vector to each pixel in the image.
- These velocities are **induced** by the **relative motion** between the camera and the 3D scene.
- The **MF** can be thought as the *projection* of the 3D velocities on the image plane.
Motion Field and Optical Flow Field

- Motion field: projection of 3D motion vectors on image plane

\[ \mathbf{v}_0 = \frac{d\mathbf{r}_0}{dt}, \quad \mathbf{v}_1 = \frac{d\mathbf{r}_i}{dt} \]

Object point \( P_0 \) has velocity \( \mathbf{v}_0 \), induces \( \mathbf{v}_i \) in image

- Optical flow field: apparent motion of brightness patterns
- We equate motion field with optical flow field

\[ \mathbf{r}_0 \text{ related to } \mathbf{r}_i \text{ by } \frac{\mathbf{r}_i}{\mathbf{f}} = \frac{\mathbf{r}_0}{\mathbf{r}_0 \cdot \hat{z}_0} \]
2 Cases Where this Assumption Clearly is not Valid

(a) A smooth sphere is rotating under constant illumination. Thus the optical flow field is zero, but the motion field is not.

(b) A fixed sphere is illuminated by a moving source—the shading of the image changes. Thus the motion field is zero, but the optical flow field is not.
What is Meant by Apparent Motion of Brightness Pattern?

The apparent motion of brightness patterns is an awkward concept. It is not easy to decide which point $P'$ on a contour $C'$ of constant brightness in the second image corresponds to a particular point $P$ on the corresponding contour $C$ in the first image.
The aperture problem
Aperture Problem

(a) Line feature observed through a small aperture at time $t$.

(b) At time $t+\delta t$ the feature has moved to a new position. It is not possible to determine exactly where each point has moved. From local image measurements only the flow component perpendicular to the line feature can be computed.

Normal flow: Component of flow perpendicular to line feature.
Brightness Constancy Equation

- Let P be a moving point in 3D:
  - At time $t$, P has coords $(X(t), Y(t), Z(t))$
  - Let $p=(x(t),y(t))$ be the coords. of its image at time $t$.
  - Let $E(x(t),y(t),t)$ be the brightness at $p$ at time $t$.

- **Brightness Constancy Assumption:**
  - As P moves over time, $E(x(t),y(t),t)$ remains constant.
Brightness Constraint Equation

Let \( E(x, y, t) \) be the irradiance and \( u(x, y), v(x, y) \) the components of optical flow.

\[
E(x + u\delta t, y + v\delta t, t + \delta t) = E(x, y, t)
\]

Taylor expansion

\[
E(x, y, t) + \delta x \frac{\partial E}{\partial x} + \delta y \frac{\partial E}{\partial y} + \delta t \frac{\partial E}{\partial t} + e = E(x, y, t)
\]

dividing by \( \delta t \) and taking limit \( \delta t \to 0 \)

\[
\frac{\partial E}{\partial x} \frac{dx}{dt} + \frac{\partial E}{\partial y} \frac{dy}{dt} + \frac{\partial E}{\partial t} = 0
\]

which is the expansion of the total derivative

\[
\frac{dE}{dt} = 0
\]

short: \[
E_x u + E_y v + E_t = 0
\]
Brightness Constancy Equation

\[ E(x(t), y(t), t) = \text{Constant} \]

Taking derivative \text{wrt} time:

\[
\frac{dE(x(t), y(t), t)}{dt} = 0
\]

\[
\frac{\partial E}{\partial x} \frac{dx}{dt} + \frac{\partial E}{\partial y} \frac{dy}{dt} + \frac{\partial E}{\partial t} = 0
\]
Brightness Constancy Equation

\[
\frac{\partial E}{\partial x} \frac{dx}{dt} + \frac{\partial E}{\partial y} \frac{dy}{dt} + \frac{\partial E}{\partial t} = 0
\]

Let

\[ \nabla E = \begin{bmatrix} \frac{\partial E}{\partial x} \\ \frac{\partial E}{\partial y} \end{bmatrix} \] \hspace{1cm} \text{(Frame spatial gradient)}

\[ \nu = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dt}{dt} \end{bmatrix} \] \hspace{1cm} \text{(optical flow)}

and

\[ E_t = \frac{\partial E}{\partial t} \] \hspace{1cm} \text{(derivative across frames)}
Brightness Constancy Equation

\[
\frac{\partial E}{\partial x} \frac{dx}{dt} + \frac{\partial E}{\partial y} \frac{dy}{dt} + \frac{\partial E}{\partial t} = 0
\]

Becomes:

\[
(\nabla E)^T \cdot \nu + E_t = 0
\]

The OF is CONSTRAINED to be on a line!
Interpretation

Values of $(u, v)$ satisfying the constraint equation lie on a straight line in velocity space. A local measurement only provides this constraint line (aperture problem).

Normal flow $u_n$

$$(E_x, E_y) \cdot (u, v) = -E_t$$

Let $n = \frac{(E_x, E_y)^T}{\left\| (E_x, E_y)^T \right\|}$

$$u_n = (u \cdot n)n = \begin{pmatrix} -\frac{E_x E_t}{E_x^2 + E_y^2}, -\frac{E_y E_t}{E_x^2 + E_y^2} \end{pmatrix}^T$$
Optical flow equation

\[ 0 = I_t + \nabla I \cdot [u \ v] \]

Barber Pole illusion
http://www.sandlotscience.com/Ambiguous/barberpole.htm
Solving the aperture problem

- How to get more equations for a pixel?
  - Basic idea: impose additional constraints
    - most common is to assume that the flow field is smooth locally
    - one method: pretend the pixel’s neighbors have the same \((u,v)\)
      - If we use a 5x5 window, that gives us 25 equations per pixel!

\[
0 = I_t(p_i) + \nabla I(p_i) \cdot [u \ v] \\
\begin{bmatrix}
  I_x(p_1) & I_y(p_1) \\
  I_x(p_2) & I_y(p_2) \\
  \vdots & \vdots \\
  I_x(p_{25}) & I_y(p_{25})
\end{bmatrix} 
\begin{bmatrix}
  u \\
  v
\end{bmatrix} = 
\begin{bmatrix}
  I_t(p_1) \\
  I_t(p_2) \\
  \vdots \\
  I_t(p_{25})
\end{bmatrix}
\]

\[
A \quad d \quad b
\]

25x2  2x1  25x1
Constant flow

- Prob: we have more equations than unknowns

\[
\begin{bmatrix}
A & d = b \\
25x2 & 2x1 & 25x1
\end{bmatrix} \text{ minimize } \|Ad - b\|^2
\]

- Solution: solve least squares problem

  - minimum least squares solution given by solution (in d) of:

\[
(A^TA) d = A^Tb
\]

\[
\begin{bmatrix}
\sum I_xI_x & \sum I_xI_y \\
\sum I_xI_y & \sum I_yI_y
\end{bmatrix}
\begin{bmatrix}
u \\
v
\end{bmatrix} =
-\begin{bmatrix}
\sum I_xI_t \\
\sum I_yI_t
\end{bmatrix}
\]

- The summations are over all pixels in the K x K window
Taking a closer look at \((A^T A)\)

\[
A = \begin{bmatrix}
    E_x(p_1) & E_y(p_1) \\
    E_x(p_2) & E_y(p_2) \\
    \vdots & \vdots \\
    E_x(p_{N2}) & E_y(p_{N2})
\end{bmatrix}
\]

\[
A^T = \begin{bmatrix}
    E_x(p_1) & E_x(p_2) & \cdots & E_x(p_{N2}) \\
    E_y(p_1) & E_y(p_2) & \cdots & E_y(p_{N2})
\end{bmatrix}
\]

\[
A^T A = \begin{bmatrix}
    \sum E_x^2 & \sum E_x E_y \\
    \sum E_x E_y & \sum E_y^2
\end{bmatrix}
\]

This is the same matrix we used for corner detection!
Taking a closer look at \((A^T A)\)

The matrix for corner detection:

\[
A^T A = \begin{bmatrix}
\sum E_x^2 & \sum E_x E_y \\
\sum E_x E_y & \sum E_y^2
\end{bmatrix}
\]

is singular (not invertible) when \(\det(A^T A) = 0\)

But \(\det(A^T A) = \prod \lambda_i = 0 \rightarrow\) one or both e.v. are 0

One e.v. = 0 \(\rightarrow\) no corner, just an edge

Two e.v. = 0 \(\rightarrow\) no corner, homogeneous region

\{ Aperture Problem ! \}
Edge

\[ \sum \nabla I (\nabla I)^T \]

- large gradients, all the same
- large \( \lambda_1 \), small \( \lambda_2 \)
Low texture region

\[ \sum \nabla I (\nabla I)^T \]

- gradients have small magnitude
- small \( \lambda_1 \), small \( \lambda_2 \)
High textured region

\[
\sum \nabla I (\nabla I)^T
\]

- gradients are different, large magnitudes
- large $\lambda_1$, large $\lambda_2$
An improvement …

• NOTE:
  – The assumption of constant OF is more likely to be wrong as we move away from the point of interest (the center point of Q)

Use weights to control the influence of the points: the farther from p, the less weight
Solving for $v$ with weights:

- Let $W$ be a diagonal matrix with weights
- Multiply both sides of $Av = b$ by $W$:
  \[ W A v = W b \]
- Multiply both sides of $WAv = Wb$ by $(WA)^T$:
  \[ A^T WWA v = A^T WWb \]
- $A^T W^2A$ is square (2x2):
  - $(A^T W^2A)^{-1}$ exists if $\det(A^T W^2A) \neq 0$
- Assuming that $(A^T W^2A)^{-1}$ does exists:
  \[ (A^T W^2A)^{-1} (A^T W^2A) v = (A^T W^2A)^{-1} A^T W^2b \]
  \[ v = (A^T W^2A)^{-1} A^T W^2b \]
Observation

- This is a two image problem BUT
  - Can measure sensitivity by just looking at one of the images!
  - This tells us which pixels are easy to track, which are hard
    - very useful later on when we do feature tracking...
Revisiting the small motion assumption

- Is this motion small enough?
  - Probably not—it’s much larger than one pixel ($2^{\text{nd}}$ order terms dominate)
  - How might we solve this problem?
Iterative Refinement

• **Iterative Lucas-Kanade Algorithm**
  1. Estimate velocity at each pixel by solving Lucas-Kanade equations
  2. Warp H towards I using the estimated flow field
     - *use image warping techniques*
  3. Repeat until convergence
Reduce the resolution!
Coarse-to-fine optical flow estimation
Coarse-to-fine optical flow estimation

Gaussian pyramid of image H

Gaussian pyramid of image I

run iterative L-K

warp & upsample

run iterative L-K
Optical flow result
Additional Constraints

- Additional constraints are necessary to estimate optical flow, for example, constraints on size of derivatives, or parametric models of the velocity field.
- Horn and Schunck (1981): global smoothness term

\[ e_s = \iint_D (u_x^2 + u_y^2) + (v_x^2 + v_y^2) \, dx \, dy : \text{departure from smoothness} \]

\[ e_c = \iint_D (E_x u + E_y v + E_t)^2 \, dx \, dy : \text{error in optical flow constraint equation} \]

Let \( \nabla A = (A_x, A_y)^T \) denote the gradient of \( A \)

\[ \iint (\nabla E \cdot u + E_t)^2 + \lambda (\|\nabla u\|_2^2 + \|\nabla v\|_2^2) \, dx \, dy \to \min \]

- This approach is called regularization.
- Solve by means of calculus of variation.
Discrete implementation leads to iterative equations

\[ u^{n+1} = \bar{u}^n - \left( \frac{E_x \bar{u}^n + E_y \bar{v}^n + E_t}{1 + \frac{E_x^2 + E_y^2}{\lambda}} \right) E_x \]

\[ v^{n+1} = \bar{v}^n - \left( \frac{E_x \bar{u}^n + E_y \bar{v}^n + E_t}{1 + \frac{E_x^2 + E_y^2}{\lambda}} \right) E_y \]

\( \bar{u}, \bar{v} \) denotes local averages of \( u \) and \( v \)

In the iterative scheme for estimating the optical flow, the new value \((u, v)\) at a point is the average of the values of the neighbors \((\bar{u}, \bar{v})\), minus an adjustment in the direction toward the constraint line.
Other Differential Techniques

- Lucas Kanade (1984): Weighted least-squares (LS) fit to a constant model of $u$ in a small neighborhood $\Omega$;

$$\sum_{x \in \Omega} W^2(x)(\nabla E(x,t) \cdot u + E_t(x,t))^2 \rightarrow \min$$

Denote $A = (\nabla E(x_1), \ldots, \nabla E(x_n))^T$, $W = \text{diag}(W(x_1), \ldots, W(x_n))^T$, $b = -(E_t(x_1), \ldots, E_t(x_n))^T$

$$u = (A^T W^2 A)^{-1} A^T W^2 b$$

- Nagel (1983,87): Oriented smoothness constraint; smoothness is not imposed across edges

$$\int \int (\nabla E^T u + E_t)^2 + \frac{\alpha^2}{\|\nabla E\|^2 + 2\delta} \times \left[ (u_x E_y - u_y E_x)^2 + (v_x E_y - v_y E_x)^2 + \delta (\|u\|^2 + \|v\|^2) \right]$$

- Uras et al. (1988): Use constraints on second-order derivatives

$$\frac{d\nabla E(x,t)}{dt} = 0 \quad \begin{bmatrix} E_{xx}(x,t) & E_{xy}(x,t) \\ E_{yx}(x,t) & E_{yy}(x,t) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} E_{tx}(x,t) \\ E_{ty}(x,t) \end{bmatrix}$$
Classification of Optical Flow Techniques

- Gradient-based methods
- Frequency-domain methods
- Correlation methods
3 Computational Stages

1. Prefiltering or smoothing with low-pass/band-pass filters to enhance signal-to-noise ratio
2. Extraction of basic measurements (e.g., spatiotemporal derivatives, spatiotemporal frequencies, local correlation surfaces)
3. Integration of these measurements, to produce 2D image flow using smoothness assumptions
Energy-based Methods

  Fourier transform of a translating 2D pattern:

\[ FE(w_x, w_y, w_t) = FE(w_x, w_y, 0) \delta(w_v + w_y + w_t) \]

All the energy lies on a plane through the origin in frequency space
Local energy is extracted using velocity-tuned filters (for example, Gabor-energy filters)
Motion is found by fitting the best plane in frequency space

- Fleet Jepson (1990): Phase-based Technique
  - Assumption that phase is preserved (as opposed to amplitude)
  - Velocity tuned band pass filters have complex-valued outputs

\[ R(x, t, w) = \rho(x, t, w)e^{i\phi(x, t, w)} \]

with \( \rho \) the amplitude and \( \phi \) the phase

\[ \frac{d\phi}{dt} = 0 \quad \text{or} \quad \phi_x u + \phi_y v + \phi_t = 0 \]
Correlation-based Methods

Anandan (1987), Singh (1990)

1. Find displacement \((dx, dy)\) which maximizes cross correlation or minimizes sum of squared differences (SSD)

\[
CC(dx, dy) = \sum_{i=-n}^{+n} \sum_{j=-n}^{+n} W(i, j) \cdot E_1(i, j) \cdot E_2(i - dx, j - dy)
\]

2. Smooth the correlation outputs

\[
SSD(dx, dy) = \sum_{j=-n}^{+n} \sum_{i=-n}^{+n} W(i, j) \cdot (E_1(i, j) - E_2(i - dx, j - dy))^2
\]
Sources:

- Horn (1986)
- http://www.cfar.umd.edu/~fer/postscript/ouchipapernew.ps.gz (paper on Ouchi illusion)
- http://www.cis.upenn.edu/~beau/home.html
  http://www.isi.uu.nl/people/michael/of.html (code for optical flow estimation techniques)