**Homework 1**

I. **Uncertainty Principle and the Time-Bandwidth Product**

1) Interpret the time-bandwidth product inequality in terms of space and frequency localization requirements described in the Marr-Hildreth paper.

The time-bandwidth inequality in terms of the 2D filters states that spatial localization and frequency localization are conflicting requirements – narrow spatial support causes a large spread in the frequency domain whereas a wide spatial support causes narrowing of the frequency spectrum.

2) Prove that the Gaussian function attains the lower-bound.

Let \( f(t) = \frac{1}{\sqrt{2\pi} \sigma} \exp(-t^2 / 2\sigma^2) \). Then, \( E = \int_{-\infty}^{+\infty} |f(t)|^2 \, dt = \frac{1}{\sqrt{2\pi}^2} \int_{-\infty}^{+\infty} \exp(-t^2 / 2\sigma^2) \, dt \).

Rearrange the terms so that the integral appears as the integral of a pdf.

\[
E = \frac{1}{\sqrt{2\pi}^2} \frac{1}{\sqrt{2\pi} / \sqrt{2}} \frac{1}{\sqrt{2}} \int_{-\infty}^{+\infty} \exp(-t^2 / 2(\sigma^2 / 2)) \, dt
\]

Now,

\[
(\Delta t)^2 = \frac{1}{E} \int_{-\infty}^{+\infty} t^2 |f(t)|^2 \, dt
\]

\[
= \frac{1}{E} \frac{1}{\sqrt{2\pi}^2} \frac{1}{\sqrt{2\pi} / \sqrt{2}} \frac{1}{\sqrt{2}} \int_{-\infty}^{+\infty} t^2 \exp(-t^2 / 2(\sigma^2 / 2)) \, dt
\]

\[
= \frac{1}{E} \frac{1}{\sqrt{2\pi}^2} \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \frac{1}{2} \sigma^2
\]

The Fourier transform of \( F(\omega) \) is given by \( F(\omega) = \exp(-\sigma^2 \omega^2 / 2) \). Similar to the above computations, it can be shown that

\[
(\Delta \omega)^2 = \frac{1}{2\pi E} \int_{-\infty}^{+\infty} \omega^2 |F(\omega)|^2 \, d\omega
\]

\[
= \frac{1}{2\pi} \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \omega^2 \exp(-\omega^2 / 2(1/(2\sigma^2))) \, d\omega
\]

\[
= \frac{1}{2\pi} \frac{\sqrt{2\pi}}{\sqrt{2\pi}} V ar(\omega)
\]

\[
= \frac{1}{2\pi} \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \frac{1}{2\sigma^2}
\]

Hence, for the Gaussian function

\[
(\Delta t)^2 (\Delta \omega)^2 = 1/4
\]

\[
\Rightarrow \Delta t \Delta \omega = 1/2
\]

II. **Optimizing Detection and Localization**

\[
SNR = \frac{\left| \int_{-W}^{+W} G(-x)f(x) \, dx \right|}{n_0 \sqrt{\int_{-W}^{+W} f(x)^2 \, dx}} \quad \text{Localization} = \frac{\left| \int_{-W}^{+W} G'(-x)f'(x) \, dx \right|}{n_0 \sqrt{\int_{-W}^{+W} f'(x)^2 \, dx}}
\]
Questions

1) Using the Cauchy-Schwarz inequality for integrals, show that the upper bounds for SNR and Localization are $n_0^{-1} \sqrt{\int_{-W}^{+W} G^2(x)dx}$ and $n_0^{-1} \sqrt{\int_{-W}^{+W} G'^2(x)dx}$ respectively. ($G(x)$ is either symmetric or anti-symmetric.)

The Cauchy-Schwarz inequality for integrals states that given two square-integrable functions $p(x), q(x)$,

$$\left| \int p(x)q(x)dx \right|^2 \leq \int |p(x)|^2 dx \int |q(x)|^2 dx \quad (16)$$

with equality occurring when $p(x) = kq(x)$. Substituting, $p(x) = G(-x)$ and $q(x) = f(x)$, we get

$$\left| \int G(-x)f(x)dx \right|^2 \leq \int |G(-x)|^2 dx \int |f(x)|^2 dx \quad (17)$$

$$\Rightarrow \frac{\left| \int G(-x)f(x)dx \right|}{n_0 \sqrt{\int |f(x)|^2 dx}} \leq n_0^{-1} \sqrt{\int |G(-x)|^2 dx} \quad (18)$$

Since, $f(x), G(x)$ are real signals, the modulus sign can be removed from the above. This proves the bound for the SNR. Similarly, the bound for Localization can be derived.

2) Then, show that $f(x) = G(-x)$ attains these upper bounds.

By the Cauchy-Schwarz inequality the equality is attained when $f(x) = kG(-x)$, for all real $k$, hence $f(x) = G(-x)$ attains these upper bounds.