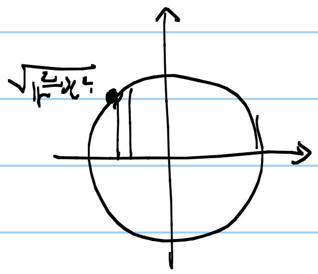
4.5.3, 4.6.7, 4.6.10, and 4.7.10

4.5.3. $X^2+Y^2 \leq r^2$

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{\pi r^2} & x^2+y^2 \leq r^2 \\ 0 & \text{o.w.} \end{cases}$$

(a). $f_X(x) = ?$

$$f_X(x) = \int_Y f_{X,Y}(x,y)\, dy.$$



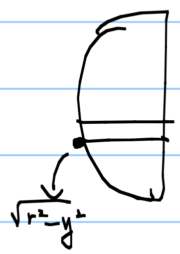
$$= 2\int_0^{\sqrt{r^2-x^2}} \frac{1}{\pi r^2}\, dy.$$

$$= \frac{2}{\pi r^2}\sqrt{r^2-x^2}.$$

(b). $f_Y(y) = ?$

$$f_Y(y) = \int_X f_{X,Y}(x,y)\, dx$$

$$= \frac{2}{\pi r^2}\sqrt{r^2-y^2}$$

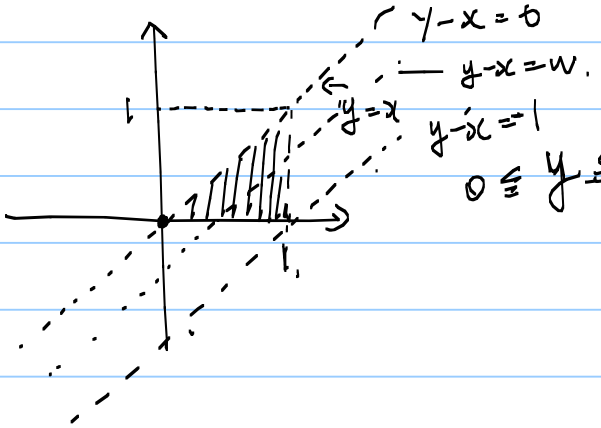


4.6.7.

$$f_{X,Y}(x,y) = \begin{cases} 6y & 0 \le y \le x \le 1. \\ 0 & \text{o.w.} \end{cases}$$

$W = Y - X.$

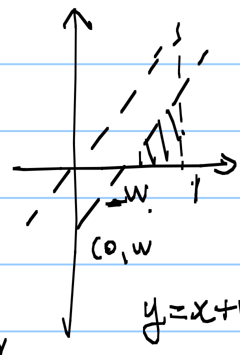
(a) what is the range of W.



$0 \le y \le x \le 1.$

$-1 \le W \le 0.$

(b). $F_W(w)$, $f_W(w)$.



$y = x + w.$

$$F_W(w) = P[W \le w]$$

$$= P[Y - X \le w].$$

$$= \int_{-w}^{1} \int_{0}^{x+w} f_{X,Y}(x,y)\,dy\,dx.$$

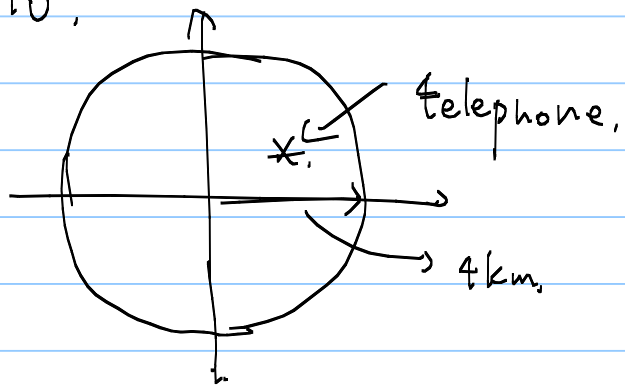
$$= \int_{-w}^{1} \int_{0}^{x+w} 6y\,dy\,dx$$

$$= (1+w)^3$$

$$\begin{cases} 0 & w < -1 \\ (1+w)^3 & -1 \le w \le 0 \\ 1 & w > 0 \end{cases}$$

$$f_W(w) = \frac{dF_W(w)}{dw} = \begin{cases} 3(1+w)^2 & -1 \le w \le 0 \\ 0 & \text{o.w.} \end{cases}$$

4.6.10.

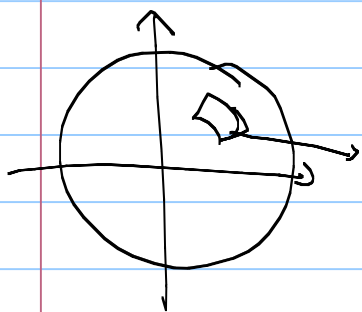


$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{16\pi}, & x^2+y^2 \le 16 \\ 0 & o.w. \end{cases}$$

$F_R(r) = P[$telephones are within $R]$.

$$R = X^2 + Y^2$$

$$= P[R \le r] = P[X^2+Y^2 \le r]$$



$$= \int_0^{2\pi}\int_0^r \frac{r'}{16\pi}\, dr'\, d\theta = \frac{r^2}{16}$$

$$dR = r\, dr\, d\theta$$

$$F_R(r) = \begin{cases} 0 & r < 0 \\ \frac{r^2}{16} & 0 \le r \le 4 \\ 1 & r > 4 \end{cases}$$

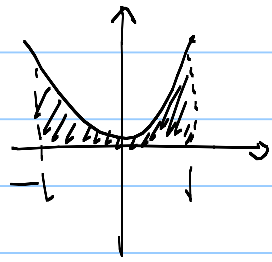
$$f_r(r) = \begin{cases} \frac{r}{8} & 0 \le r \le r \\ 0 & o.w \end{cases}$$

4.7.10.

$$f_{X,Y}(x,y) = \begin{cases} \frac{5x^2}{2} & -1 \leq x \leq 1, \\ & 0 \leq y \leq x^2, \\ 0 & \text{o.w.} \end{cases}$$

(a). $E[X]$ $Var[X]$.

$$f_X(x) = \int_0^{x^2} \frac{5x^2}{2} \, dy.$$



$$= \frac{5}{2}x^4.$$

$$E[X] = \int_{-1}^{1} \frac{5}{2}x^4 \times x \, dx.$$

$\hookrightarrow \frac{5}{2}x^5.$

$$= \left[\frac{5}{12}x^6\right]_{-1}^{1} = 0.$$

$$Var[X] \Rightarrow E[X^2] = \int_{-1}^{1} \frac{5}{2}x^6 \, dx = \frac{5}{\cancel{14}} \times \cancel{2} = \frac{5}{7}$$

$$\frac{5}{7} - 0 = \frac{5}{7}.$$

(b) $E[Y]$ $Var[Y]$.

$$E[Y] = \int_{-1}^{1}\int_0^{x^2} y \, f_{X,Y}(x,y) \, dy \, dx = \frac{5}{14}$$

$$E[Y^2] = \int_{-1}^{1}\int_0^{x^2} y^2 f_{X,Y}(x,y) \, dy \, dx = \frac{5}{26}$$

$$Var[Y] = 0.0576.$$

(c) $\mathrm{Cov}[X,Y]$

$$= E[XY] - \overset{0}{\overbrace{E[X]E[Y]}}$$

$$= \int_{-1}^{1}\int_{0}^{x^2} xy\,\frac{5}{2}x^2\,dx\,dy$$

$$= 0$$

(d) $E[X+Y] = E[X] + E[Y] = \frac{5}{14}$

(e) $\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}[X,Y]$

$$= \frac{5}{7} + 0.0576 = \underline{0.7719}$$