Problem 4.3.3 Solution

We recognize that the given joint PMF is written as the product of two marginal PMFs $P_N(n)$ and $P_K(k)$ where

$$P_N(n) = \sum_{k=0}^{100} P_{N,K}(n,k) = \begin{cases} \frac{100^n e^{-100}}{n!} & n = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$P_K(k) = \sum_{n=0}^{\infty} P_{N,K}(n,k) = \begin{cases} \binom{100}{k} p^k (1-p)^{100-k} & k = 0, 1, \ldots, 100 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Problem 4.4.2 Solution

Given the joint PDF

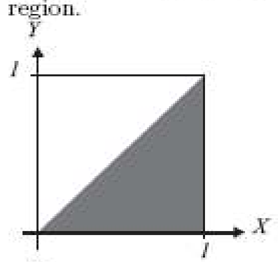
$$f_{X,Y}(x,y) = \begin{cases} cxy^2 & 0 \le x, y \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) To find the constant c integrate $f_{X,Y}(x,y)$ over the all possible values of X and Y to get

$$1 = \int_0^1 \int_0^1 cxy^2 \, dx \, dy = c/6 \tag{2}$$

Therefore $c = 6$.

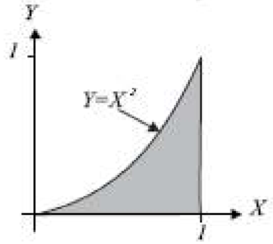
(b) The probability $P[X \ge Y]$ is the integral of the joint PDF $f_{X,Y}(x,y)$ over the indicated shaded region.



$$P[X \ge Y] = \int_0^1 \int_0^x 6xy^2 \, dy \, dx \tag{3}$$

$$= \int_0^1 2x^4 \, dx \tag{4}$$

$$= 2/5 \tag{5}$$

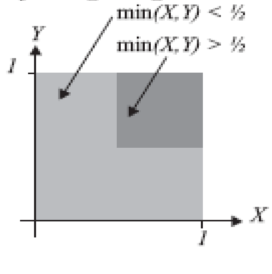


Similarly, to find $P[Y \le X^2]$ we can integrate over the region shown in the figure.

$$P\left[Y \le X^2\right] = \int_0^1 \int_0^{x^2} 6xy^2 \, dy \, dx \tag{6}$$

$$= 1/4 \tag{7}$$

(c) Here we can choose to either integrate $f_{X,Y}(x,y)$ over the lighter shaded region, which would require the evaluation of two integrals, or we can perform one integral over the darker region by recognizing

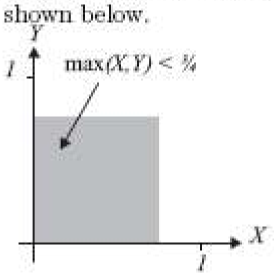


$$P[\min(X,Y) \le 1/2] = 1 - P[\min(X,Y) > 1/2] \tag{8}$$

$$= 1 - \int_{1/2}^1 \int_{1/2}^1 6xy^2 \, dx \, dy \tag{9}$$

$$= 1 - \int_{1/2}^1 \frac{9y^2}{4} \, dy = \frac{11}{32} \tag{10}$$

(d) The probability $P[\max(X,Y) \leq 3/4]$ can be found be integrating over the shaded region shown below.



$$P\left[\max(X,Y) \leq 3/4\right] = P\left[X \leq 3/4, Y \leq 3/4\right] \quad (11)$$

$$= \int_0^{\frac{3}{4}} \int_0^{\frac{3}{4}} 6xy^2 \, dx \, dy \quad (12)$$

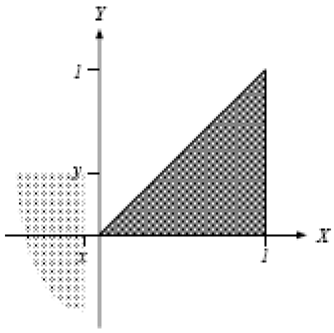
$$= \left(x^2\Big|_0^{3/4}\right)\left(y^3\Big|_0^{3/4}\right) \quad (13)$$

$$= (3/4)^5 = 0.237 \quad (14)$$

Problem 4.4.4 Solution

The only difference between this problem and Example 4.5 is that in this problem we must integrate the joint PDF over the regions to find the probabilities. Just as in Example 4.5, there are five cases. We will use variable u and v as dummy variables for x and y.

- $x < 0$ or $y < 0$

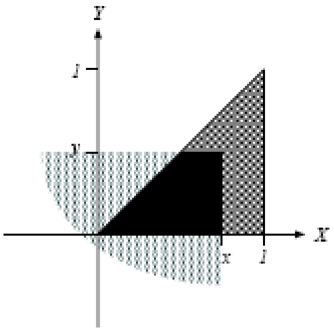


In this case, the region of integration doesn't overlap the region of nonzero probability and

$$F_{X,Y}(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(u,v) \, du \, dv = 0 \quad (1)$$

- $0 < y \leq x \leq 1$

In this case, the region where the integral has a nonzero contribution is



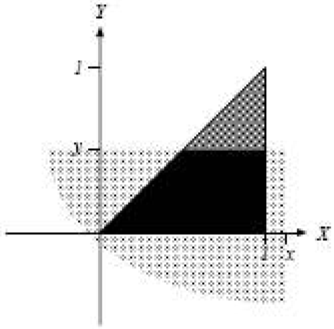
$$F_{X,Y}(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(u,v) \, dy \, dx \quad (2)$$

$$= \int_0^y \int_v^x 8uv \, du \, dv \quad (3)$$

$$= \int_0^y 4(x^2 - v^2) v \, dv \quad (4)$$

$$= 2x^2v^2 - v^4\Big|_{v=0}^{v=y} = 2x^2y^2 - y^4 \quad (5)$$

- $0 < x \le y$ and $0 \le x \le 1$

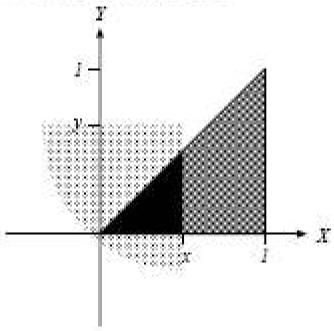


$$F_{X,Y}(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(u,v)\, dv\, du \tag{6}$$

$$= \int_{0}^{x} \int_{0}^{u} 8uv\, dv\, du \tag{7}$$

$$= \int_{0}^{x} 4u^3\, du = x^4 \tag{8}$$

- $0 < y \le 1$ and $x \ge 1$



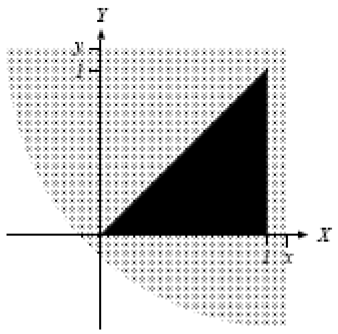
$$F_{X,Y}(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(u,v)\, dv\, du \tag{9}$$

$$= \int_{0}^{y} \int_{v}^{1} 8uv\, du\, dv \tag{10}$$

$$= \int_{0}^{y} 4v(1 - v^2)\, dv \tag{11}$$

$$= 2y^2 - y^4 \tag{12}$$

- $x \ge 1$ and $y \ge 1$



In this case, the region of integration completely covers the region of nonzero probability and

$$F_{X,Y}(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(u,v)\, du\, dv \tag{13}$$

$$= 1 \tag{14}$$

The complete answer for the joint CDF is

$$F_{X,Y}(x,y) = \begin{cases} 0 & x < 0 \text{ or } y < 0 \\ 2x^2y^2 - y^4 & 0 < y \le x \le 1 \\ x^4 & 0 \le x \le y, 0 \le x \le 1 \\ 2y^2 - y^4 & 0 \le y \le 1, x \ge 1 \\ 1 & x \ge 1, y \ge 1 \end{cases} \tag{15}$$

Problem 4.5.3 Solution
Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 1/(\pi r^2) & 0 \leq x^2 + y^2 \leq r^2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) The marginal PDF of X is

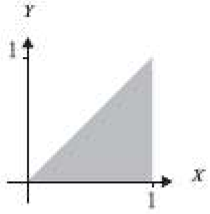
$$f_X(x) = 2\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \frac{1}{\pi r^2}\,dy = \begin{cases} \frac{2\sqrt{r^2-x^2}}{\pi r^2} & -r \leq x \leq r, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(b) Similarly, for random variable Y,

$$f_Y(y) = 2\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{1}{\pi r^2}\,dx = \begin{cases} \frac{2\sqrt{r^2-y^2}}{\pi r^2} & -r \leq y \leq r, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Problem 4.5.6 Solution

(a) The joint PDF of X and Y and the region of nonzero probability are



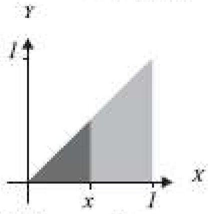
$$f_{X,Y}(x,y) = \begin{cases} cy & 0 \leq y \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b) To find the value of the constant, c, we integrate the joint PDF over all x and y.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = \int_0^1\int_0^x cy\,dy\,dx = \int_0^1 \frac{cx^2}{2}\,dx = \left.\frac{cx^3}{6}\right|_0^1 = \frac{c}{6}. \tag{2}$$

Thus $c = 6$.

(c) We can find the CDF $F_X(x) = P[X \leq x]$ by integrating the joint PDF over the event $X \leq x$. For $x < 0$, $F_X(x) = 0$. For $x > 1$, $F_X(x) = 1$. For $0 \leq x \leq 1$,



$$F_X(x) = \iint_{x' \leq x} f_{X,Y}(x',y')\,dy'\,dx' \tag{3}$$

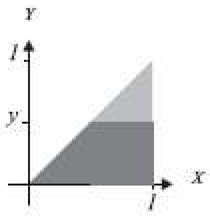
$$= \int_0^x\int_0^{x'} 6y'\,dy'\,dx' \tag{4}$$

$$= \int_0^x 3(x')^2\,dx' = x^3. \tag{5}$$

The complete expression for the joint CDF is

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x^3 & 0 \leq x \leq 1 \\ 1 & x \geq 1 \end{cases} \tag{6}$$

(d) Similarly, we find the CDF of Y by integrating $f_{X,Y}(x,y)$ over the event $Y \leq y$. For $y < 0$, $F_Y(y) = 0$ and for $y > 1$, $F_Y(y) = 1$. For $0 \leq y \leq 1$,



$$F_Y(y) = \iint_{y' \leq y} f_{X,Y}(x', y')\, dy'\, dx' \tag{7}$$

$$= \int_0^y \int_{y'}^1 6y'\, dx'\, dy' \tag{8}$$

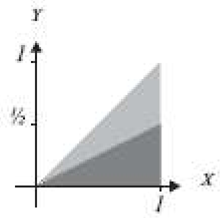
$$= \int_0^y 6y'(1 - y')\, dy' \tag{9}$$

$$= 3(y')^2 - 2(y')^3\big|_0^y = 3y^2 - 2y^3. \tag{10}$$

The complete expression for the CDF of Y is

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ 3y^2 - 2y^3 & 0 \leq y \leq 1 \\ 1 & y > 1 \end{cases} \tag{11}$$

(e) To find $P[Y \leq X/2]$, we integrate the joint PDF $f_{X,Y}(x,y)$ over the region $y \leq x/2$.



$$P[Y \leq X/2] = \int_0^1 \int_0^{x/2} 6y\, dy\, dx \tag{12}$$

$$= \int_0^1 3y^2\big|_0^{x/2}\, dx \tag{13}$$

$$= \int_0^1 \frac{3x^2}{4}\, dx = 1/4 \tag{14}$$