ENEE 324 – Spring '09

Solution For HW #6

Problem 2.7.2 Solution

Whether a lottery ticket is a winner is a Bernoulli trial with a success probability of 0.001. If we buy one every day for 50 years for a total of $50 \cdot 365 = 18250$ tickets, then the number of winning tickets T is a binomial random variable with mean

$$E[T] = 18250(0.001) = 18.25 \tag{1}$$

Since each winning ticket grosses \$1000, the revenue we collect over 50 years is $R = 1000T$ dollars. The expected revenue is

$$E[R] = 1000E[T] = 18250 \tag{2}$$

But buying a lottery ticket everyday for 50 years, at \$2.00 a pop isn't cheap and will cost us a total of $18250 \cdot 2 = \$36500$. Our net profit is then $Q = R - 36500$ and the result of our loyal 50 year patronage of the lottery system, is disappointing expected loss of

$$E[Q] = E[R] - 36500 = -18250 \tag{3}$$

Problem 2.7.7 Solution

We define random variable W such that $W = 1$ if the circuit works or $W = 0$ if the circuit is defective. (In the probability literature, W is called an indicator random variable.) Let R_s denote the profit on a circuit with standard devices. Let R_u denote the profit on a circuit with ultrareliable devices. We will compare $E[R_s]$ and $E[R_u]$ to decide which circuit implementation offers the highest expected profit.

The circuit with standard devices works with probability $(1-q)^{10}$ and generates revenue of k dollars if all of its 10 constituent devices work. We observe that we can we can express R_s as a function $r_s(W)$ and that we can find the PMF $P_W(w)$:

$$R_s = r_s(W) = \begin{cases} -10 & W = 0, \\ k - 10 & W = 1, \end{cases} \qquad P_W(w) = \begin{cases} 1-(1-q)^{10} & w = 0, \\ (1-q)^{10} & w = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Thus we can express the expected profit as

$$\begin{aligned} E[r_s(W)] &= \sum_{w=0}^{1} P_W(w)\, r_s(w) && (2)\\ &= P_W(0)(-10) + P_W(1)(k-10) && (3)\\ &= (1-(1-q)^{10})(-10) + (1-q)^{10}(k-10) = (0.9)^{10}k - 10. && (4) \end{aligned}$$

For the ultra-reliable case,

$$R_u = r_u(W) = \begin{cases} -30 & W = 0, \\ k - 30 & W = 1, \end{cases} \qquad P_W(w) = \begin{cases} 1 - (1 - q/2)^{10} & w = 0, \\ (1 - q/2)^{10} & w = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Thus we can express the expected profit as

$$E[r_u(W)] = \sum_{w=0}^{1} P_W(w)\, r_u(w) \tag{6}$$

$$= P_W(0)(-30) + P_W(1)(k - 30) \tag{7}$$

$$= (1 - (1 - q/2)^{10})(-30) + (1 - q/2)^{10}(k - 30) = (0.95)^{10}k - 30 \tag{8}$$

To determine which implementation generates the most profit, we solve $E[R_u] \geq E[R_s]$, yielding $k \geq 20/[(0.95)^{10} - (0.9)^{10}] = 80.21$. So for $k < \$80.21$ using all standard devices results in greater revenue, while for $k > \$80.21$ more revenue will be generated by implementing all ultra-reliable devices. That is, when the price commanded for a working circuit is sufficiently high, we should build more-expensive higher-reliability circuits.

If you have read ahead to Section 2.9 and learned about conditional expected values, you might prefer the following solution. If not, you might want to come back and review this alternate approach after reading Section 2.9.

Let W denote the event that a circuit works. The circuit works and generates revenue of k dollars if all of its 10 constituent devices work. For each implementation, standard or ultra-reliable, let R denote the profit on a device. We can express the expected profit as

$$E[R] = P[W]\, E[R|W] + P[W^c]\, E[R|W^c] \tag{9}$$

Let's first consider the case when only standard devices are used. In this case, a circuit works with probability $P[W] = (1-q)^{10}$. The profit made on a working device is $k - 10$ dollars while a nonworking circuit has a profit of -10 dollars. That is, $E[R|W] = k - 10$ and $E[R|W^c] = -10$. Of course, a negative profit is actually a loss. Using R_s to denote the profit using standard circuits, the expected profit is

$$E[R_s] = (1-q)^{10}(k - 10) + (1 - (1-q)^{10})(-10) = (0.9)^{10}k - 10 \tag{10}$$

And for the ultra-reliable case, the circuit works with probability $P[W] = (1-q/2)^{10}$. The profit per working circuit is $E[R|W] = k - 30$ dollars while the profit for a nonworking circuit is $E[R|W^c] = -30$ dollars. The expected profit is

$$E[R_u] = (1 - q/2)^{10}(k - 30) + (1 - (1 - q/2)^{10})(-30) = (0.95)^{10}k - 30 \tag{11}$$

Not surprisingly, we get the same answers for $E[R_u]$ and $E[R_s]$ as in the first solution by performing essentially the same calculations. it should be apparent that indicator random variable W in the first solution indicates the occurrence of the conditioning event W in the second solution. That is, indicators are a way to track conditioning events.

Problem 2.8.5 Solution

(a) The expected value of X is

$$E[X] = \sum_{x=0}^{4} x P_X(x) = 0\binom{4}{0}\frac{1}{2^4} + 1\binom{4}{1}\frac{1}{2^4} + 2\binom{4}{2}\frac{1}{2^4} + 3\binom{4}{3}\frac{1}{2^4} + 4\binom{4}{4}\frac{1}{2^4} \tag{1}$$

$$= [4 + 12 + 12 + 4]/2^4 = 2 \tag{2}$$

The expected value of X^2 is

$$E\left[X^2\right] = \sum_{x=0}^{4} x^2 P_X(x) = 0^2\binom{4}{0}\frac{1}{2^4} + 1^2\binom{4}{1}\frac{1}{2^4} + 2^2\binom{4}{2}\frac{1}{2^4} + 3^2\binom{4}{3}\frac{1}{2^4} + 4^2\binom{4}{4}\frac{1}{2^4} \tag{3}$$

$$= [4 + 24 + 36 + 16]/2^4 = 5 \tag{4}$$

The variance of X is

$$\text{Var}[X] = E\left[X^2\right] - (E[X])^2 = 5 - 2^2 = 1 \tag{5}$$

Thus, X has standard deviation $\sigma_X = \sqrt{\text{Var}[X]} = 1$.

(b) The probability that X is within one standard deviation of its expected value is

$$P[\mu_X - \sigma_X \leq X \leq \mu_X + \sigma_X] = P[2 - 1 \leq X \leq 2 + 1] = P[1 \leq X \leq 3] \tag{6}$$

This calculation is easy using the PMF of X.

$$P[1 \leq X \leq 3] = P_X(1) + P_X(2) + P_X(3) = 7/8 \tag{7}$$

Problem 2.8.8 Solution

Given the following description of the random variable Y,

$$Y = \frac{1}{\sigma_x}(X - \mu_X) \tag{1}$$

we can use the linearity property of the expectation operator to find the mean value

$$E[Y] = \frac{E[X - \mu_X]}{\sigma_X} = \frac{E[X] - E[X]}{\sigma_X} = 0 \tag{2}$$

Using the fact that $\text{Var}[aX + b] = a^2 \text{Var}[X]$, the variance of Y is found to be

$$\text{Var}[Y] = \frac{1}{\sigma_X^2}\text{Var}[X] = 1 \tag{3}$$