Q1

Consider a perceptron with a threshold activation function. The output of the perceptron is either $+1$ or -1 depending on whether the linear combination of the input values is respectively positive or negative. We want to explore the possibility of using perceptrons to realize Boolean functions, where the truth value is represented by $+1$ and the false value is represented by -1.

- Show that the AND Boolean function of n Boolean variables can be realized by a perceptron. Start with the case n=2, and then generalize your solution.

Ans

Assume n = 2,
$W_0 = -1$
$W_1 = W_2 = 1;$
$W_0 + W_1X_1 + W_2X_2 > 0$
$-1 + X_1 + X_2 > 0$

To make this inequality above be held, both X_1 and X_2 should be 1.

For some n,
$W_0 = 1 - n$
$W_1 = W_2 = ... = W_n = 1;$
$W_0 + W_1X_1 + W_2X_2 + ... + W_nX_n > 0$
$(1 - n) + X_1 + X_2 + ... + X_n > 0$

To make this inequality above be held, all X_1, X_2, ... , X_n should be 1.

Ans

- How about the OR Boolean function of n Boolean variables?

For some n,

$W_0 = n - 1$

$W_1 = W_2 = ... = W_n = 1;$

$W_0 + W_1 X_1 + W_2 X_2 + ... + W_n X_n > 0$

$(n - 1) + X_1 + X_2 + ... + X_n > 0$

To make this inequality above NOT be held, all $X_1, X_2, ... , X_n$ should be -1. Note: the solution that n - 1 of $X_1, X_2, ... , X_n$ are -1 and the other one is 1 is not feasible because

$(n - 1) + -(n - 1) + 1 = 1 > 0.$

Ans

- Consider the exclusive OR Boolean function of two variables (output is equal to $+1$ if and only if exactly one of the variables is equal to $+1$). Can this be realized by a perceptron? Explain your answer.

Plug (-1, -1), (-1, 1), (1, -1) and (1, -1) in $W_0 + W_1 X_1 + W_2 X_2$ and we get four inequalities as below:

W_0 - W_1 - $W_2 < 0$ (1)
W_0 - W_1 + $W_2 > 0$ (2)
W_0 + W_1 - $W_2 > 0$ (3)
W_0 + W_1 + $W_2 < 0$ (4)

Addition of (1) and (4), we get $W_0 < 0$
Addition of (2) and (3), we get $W_0 > 0$
Confliction! Exclusive OR function of two variables cannot be realized by a perceptron.