Dirichlet Processes

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INTRODUCTION
Clustering as Probabilistic Inference

- GMM is a probabilistic model (unlike $K$-means)
- There are several latent variables:
  - Means
  - Assignments
  - (Variances)
Clustering as Probabilistic Inference

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  - Means
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- Before, we were doing EM
Clustering as Probabilistic Inference

- GMM is a probabilistic model (unlike \( K \)-means)
- There are several latent variables:
  - Means
  - Assignments
  - (Variances)
- Before, we were doing EM
- Today, new models and new methods
Nonparametric Clustering

- What if the number of clusters is not fixed?
- Nonparametric: can grow if data need it
- Probabilistic distribution over number of clusters
Dirichlet Process

- Distribution over distributions
- Parameterized by: $\alpha, G$
Dirichlet Process

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- You can then draw observations from $x \sim \text{DP}(\alpha, G)$. 
Defining a DP

- Break off sticks

\[
V_1, V_2, \cdots \sim_{\text{iid}} \text{Beta}(1, \alpha) \tag{1}
\]

\[
C_k \equiv V_k \prod_{j=1}^{k-1} (1 - V_j) \tag{2}
\]
Defining a DP

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\[ V_1, V_2, \ldots \sim_{\text{iid}} \text{Beta}(1, \alpha) \]  \hspace{1cm} (1)

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- Draw atoms

\[ \Phi_1, \Phi_2, \ldots \sim_{\text{iid}} G \]
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\[ \Phi_1, \Phi_2, \ldots \sim_{\text{iid}} G \]

- Merge into complete distribution

\[ \Theta = \sum_{k \in \mathbb{N}} C_k \delta_{\Phi_k} \]
Properties of a DPMM

- Expected value is the same as base distribution
  \[ E_{DP(\alpha,G)}[x] = E_G[x] \]  
  (3)

- As $\alpha \to \infty$, $DP(\alpha, G) = G$

- Number of components unbounded

- Impossible to represent fully on computer (truncation)

- You can nest DPs
Effect of scaling parameter $\alpha$
DP as mixture Model
The Chinese Restaurant as a Distribution

To generate an observation, you first sit down at a table. You sit down at a table proportional to the number of people sitting at the table.
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\[
x \sim \mu_1 \\
\frac{2}{7}
\]

\[
x \sim \mu_2 \\
\frac{3}{7}
\]

\[
x \sim \mu_3 \\
\frac{2}{7}
\]
The Chinese Restaurant as a Distribution

To generate an observation, you first sit down at a table. You sit down at a table proportional to the number of people sitting at the table.

\[ x \sim \mu_1 \quad x \sim \mu_2 \quad x \sim \mu_3 \]

But this is just Maximum Likelihood

Why are we talking about Chinese Restaurants?
Always can squeeze in one more table . . .

- The posterior of a DP is CRP
- A new observation has a new table / cluster with probability proportional to $\alpha$
- But this must be balanced against the probability of an observation given a cluster

$$\Theta = \sum_{k \in \mathbb{N}} C_k \delta_{\Phi_k}$$
Gibbs Sampling

- We want to know the cluster assignment of each observation
- Take a random guess initially
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- Take a random guess initially
- This provides a mean for each cluster
- Let the number of clusters grow
Gibbs Sampling

- We want to know $\tilde{z}$
- Compute $p(z_i | z_1 \ldots z_{i-1}, z_{i+1}, \ldots z_m, x, \alpha, G)$
- Update $z_i$ by sampling from that distribution
- Keep going . . .
Gibbs Sampling

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- Compute $p(z_i | z_1 \ldots z_{i-1}, z_{i+1}, \ldots z_m, x, \alpha, G)$
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Notation

\[
p(z_i = k | z_{-i}) \equiv p(z_i | z_1 \ldots z_{i-1}, z_{i+1}, \ldots z_m)
\] (4)
Gibbs Sampling for DPMM

\[ p(z_i = k \mid \tilde{z}_{-i}, \tilde{x}, \{\theta_k\}, \alpha) \]  

(5) (6)
Gibbs Sampling for DPMM

\[
p(z_i = k | \tilde{z}_{-i}, \tilde{x}, \{ \theta_k \}, \alpha) = p(z_i = k | \tilde{z}_{-i}, x_i, \tilde{x}, \theta_k, \alpha)
\]  

Dropping irrelevant terms
Gibbs Sampling for DPMM

\[ p(z_i = k | \hat{z}_{-i}, \hat{x}, \{ \theta_k \}, \alpha) \]  
\[ = p(z_i = k | \hat{z}_{-i}, x_i, \hat{x}, \theta_k, \alpha) \]  
\[ = p(z_i = k | \hat{z}_{-i}, \alpha) p(x_i | \theta_k, \hat{x}) \]

Chain rule
Gibbs Sampling for DPMM

\begin{align*}
p(z_i = k \mid \tilde{z}_i, \tilde{x}, \{\theta_k\}, \alpha) \quad & \quad (5) \\
= p(z_i = k \mid \tilde{z}_i, x_i, \tilde{x}, \theta_k, \alpha) \quad & \quad (6) \\
= p(z_i = k \mid \tilde{z}_i, \alpha) p(x_i \mid \theta_k, \tilde{x}) \quad & \quad (7) \\
= \begin{cases} 
\left( \frac{n_k}{n + \alpha} \right) \int_{\theta} p(x_i \mid \theta) p(\theta \mid G, \tilde{x}) & \text{existing} \\
\frac{\alpha}{n + \alpha} \int_{\theta} p(x_i \mid \theta) p(\theta \mid G) & \text{new} 
\end{cases} \quad & \quad (8)
\end{align*}

Applying CRP
Gibbs Sampling for DPMM

\[ p(z_i = k \mid \tilde{z}_{-i}, \tilde{x}, \{ \theta_k \}, \alpha) \]
\[ = p(z_i = k \mid \tilde{z}_{-i}, x_i, \tilde{x}, \theta_k, \alpha) \]
\[ = p(z_i = k \mid \tilde{z}_{-i}, \alpha)p(x_i \mid \theta_k, \tilde{x}) \]
\[ = \begin{cases} \left( \frac{n_k}{n + \alpha} \right) \int_\theta p(x_i \mid \theta)p(\theta \mid G, \tilde{x}) & \text{existing} \\ \frac{\alpha}{n + \alpha} \int_\theta p(x_i \mid \theta)p(\theta \mid G) & \text{new} \end{cases} \]
\[ = \begin{cases} \left( \frac{n_k}{n + \alpha} \right) \mathcal{N}(x, \frac{n\tilde{x}}{n+1}, 1) & \text{existing} \\ \frac{\alpha}{n + \alpha} \mathcal{N}(x, 0, 1) & \text{new} \end{cases} \]

Scary integrals assuming \( G \) is normal distribution with mean zero and unit variance. (Derived in optional reading.)
Algorithm for Gibbs Sampling

1. Random initial assignment to clusters
2. For iteration $i$:
   2.1 “Unassign” observation $n$
   2.2 Choose new cluster for that observation
Toy Example
Toy Example
Toy Example
Toy Example
Toy Example
Toy Example
Toy Example

New cluster created!
Toy Example
Toy Example
Toy Example
Toy Example
Toy Example
Toy Example
Toy Example
Toy Example

And repeat …
Differences between EM and Gibbs

- Gibbs often faster to implement
- EM easier to diagnose convergence
- EM can be parallelized
- Gibbs is more widely applicable
In class and next week

- Walking through DPMM clustering
- Clustering discrete data with more than one cluster per observation