Non-Monotonic Inference Rules for the Credulous Theory

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Abstract

Most of the theories that model default reasoning by means of defeasible inheritance networks proceed via construction of paths, and are therefore known as 'path-based'. In spite of many advantages, these theories also face certain well-known problems. In particular, they do poorly when cyclic nets are at issue or when it comes up to extensions. In this paper we show that these problems may be overcome by a change of environment. We begin by presenting a certain fairly standard path-based approach (similar to the credulous theory of Touretzky). Then we proceed to develop an alternative network-based formalism that relies on a set of non-monotonic inference rules and prove that both approaches map networks into the same conclusion sets. Unlike its path-based counterpart, however, the new rule-based system accommodates cyclic nets and seems to allow naturally for extensions. One such is presented: the expressive power of the system is enhanced with the idea of abnormality minimisation. As a result, we are able to say that an object is abnormal with respect to a certain default, and, thus, single out the conclusion sets that imply the least number of abnormalities. It is argued that exactly these sets are the intuitively correct ones.

1 Introduction

Default reasoning is best characterised as making sensible guesses in the context of incomplete information. It is hardly possible to overestimate its importance, for it is an essential part of our everyday reasoning patterns; and it is only natural that it has long since become an area of intensive research in the field of artificial intelligence. Many formalisms modelling default reasoning have been suggested in the last decades, and one of the more successful ones is that of defeasible inheritance networks. These nets are finite collections of nodes and positive and negative IS-A links, which we will write as \((x, z, +)\) and \((x, z, -)\). The interpretation of \((x, z, +)\) and \((x, z, -)\) depends on \(x\). If it is a general term, their intended meanings are \('x\text{-s usually are } z\text{-s}'\) and \('x\text{-s usually are not } z\text{-s}'\), respectively. Thus, in this case the links are nothing else but default statements. In case \(x\) is an
individual or a concrete object, however, \((x, z, +)\) should be read as ‘\(x\) is a \(z\)’ and \((x, z, -)\) as ‘\(x\) is not a \(z\)’. On the intuitive level the collections of IS-A links are best thought of as sets of hypotheses provided to some agent or reasoning mechanism. The primary goal of any network-based account is to link inheritance networks to their conclusion sets. Intuitively, the conclusion set of a network is the set of all and only those statements that an ideal reasoner would arrive at on the basis of the information specified in this network. Statements that are allowed to enter a conclusion set are of two kinds: positive, written as \(isa(x, z, +)\), and negative, written as \(isa(x, z, -)\). The intended meaning of the former is ‘it is natural to suppose that \(x\) is a \(z\)’ (\(x\)-s are \(z\)-s); and that of the latter is ‘it is natural to suppose that \(x\) is not a \(z\)’ (\(x\)-s are not \(z\)-s)’. In analogy to the links, we call these ‘IS-A statements’.

There are different ways to get to the conclusion set corresponding to a given network. The most widespread one is roundabout. One first establishes the so-called ‘expansion’ (expansions) of the given net, and then constructs the conclusion set from it (them). These expansions are sets of paths; and paths are but special sequences of links. For instance, the sequence consisting of \((Feathers, Bird, +)\) and \((Bird, AbleToFly, +)\), written as \((Feathers, Bird, AbleToFly, +)\), would count as a path. (Hory, [2]) suggests viewing paths as arguments supporting statements that can enter a conclusion set, and we will follow him in that. Thus, the above path would be an argument that supports \(isa(Feathers, AbleToFly, +)\). We use the term ‘path-based’ to refer to any approach that relies on paths and path sets when determining conclusion sets.

There are certain well-known problems with path-based theories: (i) being quite remote from ordinary logic, they are not convenient when it comes to extensions and (ii) they do poorly as soon as one has to deal with networks that contain cycles. Now, our main goal in this paper is to show that these problems may be overcome by transferring a certain path-based approach into a different framework. What we therefore do is this: we develop an alternative formalism that is network-, but not path-based and show that it maps nets into exactly the same conclusion sets as its path-based counterpart, but does not face the above problems.

In Section 2 we present a certain path-based account for default reasoning. It is a fairly standard system, although some of our definitions (consistency, preclusion) deviate from the received ones. After having presented the system, we turn to two problems that any path-based system necessarily has to face. In Section 3 we transfer our path-based approach into an alternative framework that falls within the tradition pioneered by Erik Sandewall in [6]: at its heart lies a set of non-monotonic inference rules that operate on statements, rather than paths. By this change of setting we immediately solve the first of the two presented problems (cycles). In Section 4 we increase the expressive power of the newly developed framework. We introduce statements
that allow us to say than an object (or kind) is abnormal with respect to a certain default statement. This enables us to compare the various possible 'lines of reasoning' on the basis of the information specified in the given net and single out the ones that implies the least number of abnormalities. We then proceed to show that it helps us to solve the second problem. This all is followed by a short concluding section and an appendix that contains the longer proofs.

2 Path-Based Approaches

An inheritance network \( \Gamma \) is defined as a finite collection of positive and negative links between nodes. \((x, y, +)\) stands for a positive and \((x, y, −)\) for a negative link from some node \(x\) to another node \(y\). We use \(s\) as a variable ranging over the signs \(+\) and \(−\), and let \(−s\) denotes the opposite sign of \(s\). A positive path from \(x_1\) to \(x_n\) through \(x_2, \ldots, x_{n−1}\), denoted by \((x_1, \sigma, x_n, +)\) or \((x_1, x_2, \ldots, x_{n−1}, x_n, +)\), is defined as a sequence of direct links \((x_1, x_2, +), (x_2, x_3, +), \ldots, (x_{n−1}, x_n, +)\). A negative path from \(x_1\) to \(x_n\) via \(x_2, \ldots, x_{n−1}\), denoted by \((x_1, \sigma, x_n, −)\) or \((x_1, x_2, \ldots, x_{n−1}, x_n, −)\), is defined as a sequence of direct links \((x_1, x_2, +), (x_2, x_3, +), \ldots, (x_{n−1}, x_n, −)\). Note that a sequence consisting of only one link is a path. Lower case Greek letters \(\pi\) through \(\tau\) are used to denote paths, while the upper case \(\Phi\) stand for an arbitrary set of paths. We say that a path of the form \((x, \sigma, y, s)\) supports the statement \(isa(x, y, s)\). The extension of this relation to path sets and sets of statements is straightforward. The notation we use is, to a large extent, due to (Simonet, [7]).

We follow (Horty [2]) in describing a pair that consists of a network \(\Gamma\) and a path set \(\Phi = (\Gamma, \Phi)\) as an epistemic context. In accordance with the received practice, we let the notion of inheritability emerge as a combination of three preliminary concepts: constructibility, conflict, and preclusion. Our definitions of the first two are standard, while that of the third one is not.

**Definition 1 (Constructibility).** A path \((x, \sigma, y, z, s)\) is constructible in the context \((\Gamma, \Phi)\) iff \((x, \sigma, y, +) \in \Phi\) and \((y, z, s) \in \Gamma\).

**Definition 2 (Conflict).** A path \((x, \sigma, y, s)\) conflicts with any path of the form \((x, \rho, y, −s)\). A path \(\pi\) is conflicted in the context \((\Gamma, \Phi)\) iff \(\Phi\) contains a path that conflicts with \(\pi\).

The definition of preclusion relies on the concept of an intermediary node:\(^1\)

\(^1\)The preclusion of (Touretzky, [8]), which is usually referred to as 'on-path preclusion', is also defined using intermediaries. However, his notion differs considerably from the present one. First, it is somewhat more stringent. Second, it is defined only for the positive paths.
Figure 1: $y$ is an intermediary to the path $(x_1, \ldots, x_k, y_1, \ldots, y_l, x_m, \ldots, x_n, +)$ in $\Phi$, for $\Phi$ contains $(x_1, \ldots, x_k, y_1, \ldots, y_l, x_m, +)$ and $y = y_i$ for some $j$, $1 \leq j \leq l$, $1 \leq k < m < n$.

**Definition 3 (Intermediary).** The node $v$ is an intermediary to the path $(x_1, \ldots, x_n, s)$ in the path set $\Phi$ if $v = x_i$ for some $1 \leq i < n$, or else $\Phi$ contains a path of the form $(x_1, \ldots, x_k, y_1, \ldots, y_l, x_m, +)$ where $1 \leq k < m < n$ and $v = y_j$ for some $1 \leq j \leq l$. (See Figure 1.)

With the intermediaries in their place, we can define preclusion:

**Definition 4 (Preclusion).** A path $(x_1, \sigma, x_n, s)$ is precluded in the context $\langle \Gamma, \Phi \rangle$ iff there is a node $v$ such that (i) $v$ is an intermediary to the path $(x_1, \sigma, x_n, +)$ in the path set $\Phi$, and (ii) $(v, x_n, -s) \in \Gamma$.

Although the question about the correct notion of preclusion is open, it is fair to say that the most wide-spread one is that of (Sandewall, [6]). It is usually defined thus:

**Definition 5 (Off-path preclusion).** A path $(x, \sigma, y, z, s)$ is precluded in the context $\langle \Gamma, \Phi \rangle$ iff there is a node $v$ such that (i) either $v = x$ or there is a path of the form $(x, \rho, v, \rho', y, +) \in \Phi$, and (ii) $(v, z, -s) \in \Gamma$.

Despite the superficial differences, the two definitions turn out to be equivalent:
Figure 2: $\Gamma_1$, the Nixon Diamond

**Proposition 1.** A path $\pi$ is precluded iff it is off-path precluded.

**Proof.** Immediate. □

The reason we opt for a non-standard definition is that it is easier to translate into the rule-based framework to which we will turn in the next section. Now we have everything we need to define inheritability, as well as the notion of a credulous expansions.

**Definition 6 (Defeasible Inheritability).**

- **Case 1:** $\pi$ is a direct link. Then $(\Gamma, \Phi)\models \pi$ iff $\pi \in \Gamma$.
- **Case 2:** $\pi$ is a compound path. Then $(\Gamma, \Phi)\models \pi$ iff
  1. $\pi$ is constructible in $(\Gamma, \Phi)$,
  2. $\pi$ is not conflicted in $(\Gamma, \Phi)$,
  3. $\pi$ is not precluded in $(\Gamma, \Phi)$.

**Definition 7 (Credulous expansion).** The path set $\Phi$ is a credulous expansion of the net $\Gamma$ iff $\Phi = \{\pi : (\Gamma, \Phi)\models \pi\}$.\(^2\)

On the intuitive level, any single credulous expansion of some given network is supposed to be one possible and internally consistent ‘line of reasoning’ on the basis of the information that this network contains. The standard example evoked to elucidate this notion is network $\Gamma_1$ that is depicted in Figure 2. Its interpretation is $n = \text{Nixon}$, $R = \text{Republican}$, $Q = \text{Quaker}$, $P = \text{pacifist}$. Now, $\Gamma_1$ has two credulous expansions. The first one consists of the paths $(n, R, +)$, $(n, Q, +)$, and $(n, Q, P, +)$. The second also contains $(n, R, +)$ and $(n, Q, +)$ (just as the first), but the place of $(n, Q, P, +)$ is taken by $(n, R, P, -)$.\(^3\)

\(^2\)Since (Horty, [2]) this way of defining expansions of (Tourezyk, [8]) has become fairly standard. It is also used, for instance, in (Dung and Son, [1]) and in (Simonet, [7]).

\(^3\)In fact, credulous expansions are called ‘credulous’ in contrast to the skeptical ones. Although the literature contains several non-equivalent definitions of the latter, what all of them share is that the skeptical ‘lines of reasoning’ are always unique.
The last two notions we need are those of network and expansion consistency. Usually the definition of network consistency goes along the lines of ‘a network is consistent if there are no two links of the form \((x, y, +)\) and \((x, y, −)\) in it’. Our definition is a generalisation of the standard one:

**Definition 8** (Network consistency). A network \(\Gamma\) is consistent if \(\Gamma\) does not contain two links of the form \((x_i, y, +)\) and \((x_j, y, −)\) such that (i) either \(x_i = x_j\), or (ii) there is a positive path of the form \((x_1, x_2, \sigma, x_n, x_1, +)\) in \(\Gamma\) with \(1 \leq i \leq j \leq n\).

Some illustrative examples of inconsistent networks are given in Figure 3. The main reason for adopting this more stringent definition results from the intuition about positive cycles. Now, a positive cycle — i.e., a positive path of the form \((x_1, x_2, \sigma, x_n, x_1, +)\) — is akin to a description of an if and only if relation between \(x_1, x_2\), etc. (cf. (Wang et al., [9])). Intuitively, if some two nodes \(x_i\) and \(x_j\) are connected by a positive cycle, all the information that is specified about \(x_i\) is just as well stated about \(x_j\), and vice versa. Given this, it only seems natural to extend the usual prohibition on the links of the form \((x, y, +)\) and \((x, y, −)\) in the way our definition does. In contrast to consistency for nets, our notion of consistency for expansions is standard. Thus we say that an expansion is consistent, if it does not contain two paths of the form \((x, \sigma, y, s)\) and \((x, \rho, y, −s)\).

The path-based approaches in general (and, thus, also the one presented here) have many nice features, but they also face certain problems. We now turn to two.

First, as soon as we have to deal with networks that contain cycles, the path-based approaches do poorly. Let us consider the simplest possible example of a (positive) cyclic net: \(\Gamma_2 = \{(A, B, +), (B, A, +)\}\). If we try to construct its credulous expansion, we soon realise that it has to be infinite. Instead of terminating after an addition of \((A, B, A, +)\) and \((B, A, B, +)\) to

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4Interestingly, a number of notorious examples of cyclic nets — nets that have no expansions (see, for instance, (Horty, [2][p. 15])) — come out inconsistent on this definition.
It keeps growing. All the longer paths are superfluous, for — supporting one or another of the same IS-A statements one of these four paths already support — they do not add anything contentwise. Now, the usual reaction to this problem is to restrict attention to acyclic nets. This, however, cannot be satisfactory, for cycles are indispensable for representing certain situations (see (Wang et al., [9][pp. 156-7])). In the next section we develop an approach that is general enough to be applicable to all kinds of networks. We will also make use of the following small lemma. What it says intuitively is that going through the same cycle again does not add any new information.

**Definition 9** (Superfluous path). Let $(\Gamma, \Phi)$ be an epistemic context and $\pi$ some path of the form $(x, \rho, y, \tau, y, +)$. We say that $\pi$ is superfluous for this context in case $\Phi$ already contains a path of the form $(x, \rho, y, +)$.

**Lemma 1.** Let $\Gamma$ be a consistent inheritance network and $\Phi$ a path set acquired from $\Gamma$. If all paths $\pi$ for which we have both $(\Gamma, \Phi) \models \pi$ and $\pi \notin \Phi$ are superfluous for $(\Gamma, \Phi)$, then $\Phi$ can be extended into an expansion $\Phi'$ of $\Gamma$.

**Proof.** A straightforward induction. □

The second problem is illustrated by network $\Gamma_3$ that is given in Figure 4. Its interpretation is as follows: $a = Alice$, $SoThr = a$ person with a sore throat, $RuN = a$ person with a running nose, $Cough = a$ person with a cough, $NasCon = a$ person with a nasal congestion, $NormTerm = a$ person with normal body temperature, and $Co = a$ person who has a cold. $\Gamma_3$ seems to be a perfectly reasonable representation of a real-life situation. Sore throat, running nose, cough, nasal congestion, and fever are the usual symptoms of the infection known as the ‘common cold’. That is to say, each of these nuisances most often is caused exactly by cold. Often enough, though, people do not develop all of the symptoms. In the particular situation we model, the individual called Alice has all the symptoms of cold but fever. Now, I argue that, despite her normal temperature, in this case it is natural to suppose that Alice has a cold.

Let us now have a look at what our path-based approach says concerning $\Gamma_3$. On it, this network has two equally significant expansions. One of them contains the paths $(a, SoThr, Co, +)$, $(a, RuN, Co, +)$, $(a, Cough, Co, +)$, and $(a, NasCon, Co, +)$; and thus supports $isa(a, Co, +)$. The other contains $(a, NormTemp, Co, −)$ and supports $isa(a, Co, −)$. Of course, the first expansion is the intuitively correct one, and we must grant that the credulous theory is able to identify it. Nevertheless, it does not provide any means for comparing different expansions, and, therefore, justifies the reasoner to infer $isa(a, Co, −)$ as much as it justifies the inference of $isa(a, Co, +)$.

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5 The various skeptical approaches do much worse, for they all unanimously invite the reasoner to remain skeptical about Alice’s having or not having a cold.
Something has to be added to the formalism for us to be able to single out the intuitively correct credulous expansion of $\Gamma_3$. Unfortunately, though, path-based approaches in general lend themselves uneasily to extensions. What we do next is this. In Section 3 we first transfer the credulous approach to a different framework — a framework that can handle cycles. In Section 4 we extend it so as to be able to handle such cases as $\Gamma_3$.

3 Non-Monotonic Inference Rules: the Basic System

The approach we develop in this section uses the framework that was pioneered by Erik Sandewall in [6]. It is a network-, but not a path-based account. At its heart lies a set of non-monotonic inference rules which operate not on paths and links, but on sets of statements. For Sandewall, each statement had to be either an atom or a negation of an atom, with every atom having one of the following forms:

- $isax(x, y, s)$,
- $isa(x, y, s)$,
- $precl(x, y, z, s)$.

We have already introduced the IS-A statements. Therefore we know that the intended meaning of $isa(x, y, s)$ is ‘it is natural to suppose that $x$ is [not] a $y$ (/x-s are [not] y-s)’. Now, statements of the $isa$-form play

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6 In fact, Sandewall also allows for statements of the form $cntr(x, y, z, s)$, but we will not use them. The purpose they served can just as well be served by the $isa$-statements.
roughly the same role in Sandewall’s framework as paths play in path-based approaches. The isax-statements, on the other hand, are more like links. In fact, any statement of the form isax \((x, y, s)\) has the same meaning as \((x, y, s)\). Thus, if \(x\) is an individual object, isax \((x, y, s)\) is a first-order statement of the form ‘\(x\) is [not] a \(y\)’; if \(x\) is a kind, it is a default statement. Finally, the meaning of the precl-statements is close to the notion of preclusion discussed above, just as the name suggests. We will read precl \((x, y, z, s)\) as ‘the default \((y, z, s)\) is precluded for \(x\)’.

The purpose of the non-monotonic inference rules is to determine the extension (or extensions)\(^7\) of a given inheritance network. These extensions are special sets of statements, and both their function and intuitive interpretation are analogous to those of credulous expansions in our path-based approach. Sandewall’s rules have the following general form: If \(D_1\) is in the set and \(D_2\) is not, then infer \(D_3\). \(D_1\), \(D_2\), and \(D_3\) stand for sets of statements. Note also that the part of the rule that is related to \(D_2\) may be left out. Extensions of a given inheritance net are determined by constructing a sequence of increasing sets of statements, \(E_0, E_1, \ldots, E_i, \ldots\), where:

\[
E_0 \text{ is the initial set of statements isax}(x, y, s) \text{ standing for the links in the net, and}
\]

each \(E_{i+1}\) is acquired from \(E_i\) by instantiating one of the inference rules, \(D_1\) has to be a subset of \(E_i\), \(D_2\) disjoint from \(E_i\), and then \(E_{i+1}\) is set to be \(E_i \cup D_3\).

This process is continued to its (possibly infinite) limit. Such a limit \(E\) is an extension, if it satisfies two conditions: (i) \(E\) is consistent, i.e., it does not contain two statements of the form \(p\) and \(\neg p\) for some \(p\), and (ii) \(E\) is a fixed point for the given set of rules, i.e., for any rule applicable to \(E\), \(D_3\) is a subset of \(E\). We should add that applying the rules in different order may result in several different extensions. Notice that our notion of consistency for extensions differs from the notions of consistency for expansions. Thus, an extension could, at least in principle, contain two statements of the form isax \((x, y, +)\) and isax \((x, y, −)\) (or isa \((x, y, +)\) and isa \((x, y, −)\)). However, this cannot happen as long as one deals with consistent inheritance nets, and in this paper we restrict our attention to such.

Unfortunately, the set of six inference rules originally proposed in (Sandewall, [6]) turned out not to be flawless. Most importantly, it gives counter-intuitive results on some fairly simple examples that are well-known in the literature. (Simonet, [7]) — the only serious follow-up to Sandewall’s framework — contains a detailed discussion of the problematic cases along with a proposal for revision of the rules. This paper, however, goes beyond a mere revision. Geneviève Simonet also presents an alternative set (or rather

\(^7\)My notion of extension corresponds to Sandewall’s notion of consistent extension.
sets) of non-monotonic inference rules drawing motivation directly from some
path-based approaches. Her basic idea is to take a certain credulous path-
based theory (which is very much alike the approach we presented in the
previous section), carry it over to Sandewall’s setting, and give a correspon-
dence proof between the two.\(^8\) However, the way this idea is implemented is
hardly satisfactory. First, just like the Sandewall rules, those of Simonet pro-
duce counter-intuitive results. Second and even more importantly, in some
concrete cases the results given by her rules differ from those suggested by
the underlying path-based approach, which cannot but cast doubts on Si-
monet’s correspondence proof. For a detailed discussion of all the problems
with (Simonet, [7]) see the B appendix of my [3]. Here we at once turn to
developing a new set of non-monotonic inference rules.

Let us first step back and compare the language of the Sandewall fram-
ework — henceforth, the rule-based framework — and that of our path-based
approach. \((x, \sigma, y, +)\) and \(isa(x, y, +)\) both carry the information that it
is natural to suppose that \(x\) is a \(y\), but there is a serious difference. The
path \((x, \sigma, y, +)\) specifies what links (defaults) ascribing \(y\) to \(x\) depends on,
while \(isa(x, y, +)\) provides no such information at all. In fact, we are simply
unable to say anything like ‘\(x\) inherits from \(y\) via \(z\)’ in Sandewall’s lan-
guage. Usually, though, this information is absolutely crucial for handling
cases of preclusion in path-based approaches, and the reader may seriously
doubt the very possibility of having the same notion of preclusion in the new
approach. Now, recall that our definition of preclusion was non-standard
(though, equivalent): instead of asking for an existence of a path of the
form \((x, \rho, v, \rho', y, +)\), we relied on the notion of an intermediary. This was
done intentionally, for the latter can be straightforwardly transferred to the
rule-based system.

Thus, we extend Sandewall’s original language by allowing for atomic
propositions of the following form: \(interm(x, y, z, s)\). One could think of
\(interm(x, y, z, s)\) as saying that node \(y\) is an intermediary to any path of
polarity \(s\) that begins with \(x\) and ends with \(z\). However, since there are no
paths in the rule-based framework, this cannot be entirely precise. A correct
though somewhat cumbersome reading of \(interm(x, y, z, s)\) would go along
the lines of ‘In case \(y\) and \(z\) justify inferring opposite statements about \(x\),
y must be given priority’. Be it as it may, \(interm\)-statements will tell us
effectively when defaults turn out precluded.

Now we can state our inference rules.

**Set \(R\) of inference rules**

1. If \(isax(x, y, s)\) is in \(E\)

   then add \(isa(x, y, s)\) and \(interm(x, x, y, s)\) to \(E\).

\(^8\)In fact, it would be more precise to talk of three different credulous theories here, for
Simonet works with three non-equivalent definitions of preclusion at the same time.
2. If \( \text{isa}(x, y, +) \), \( \text{isax}(y, z, s) \), \( \text{interm}(x, v, y, +) \), and \( \text{isax}(v, z, -s) \) are in \( E \),
then add \( \text{precl}(x, y, z, s) \) to \( E \).

3. If \( \text{isa}(x, y, +) \) and \( \text{isax}(y, z, s) \) are in \( E \),
and \( \text{precl}(x, y, z, s) \) and \( \text{isa}(x, z, -s) \) are not in \( E \),
then add \( \text{isa}(x, z, s) \), \( \neg \text{precl}(x, y, z, s) \), and \( \text{interm}(x, y, z, s) \) to \( E \).

4. If \( \text{interm}(x, y, z, +) \) and \( \text{interm}(x, z, w, s) \) are in \( E \),
then add \( \text{interm}(x, y, w, s) \) to \( E \).

Each of the rules requires a little comment. The main aim of the first one is to translate the \( \text{isax} \)-statements into the corresponding statements of the \( \text{isa} \)-form. Intuitively, if we know \( \text{isax}(x, y, s) \), \( \text{isa}(x, y, s) \) is the least we should be able to derive. This rule is carried over from (Sandewall, [6]) with the only difference that we add the statement \( \text{interm}(x, x, y, s) \) to \( E \) as well. We need it (just as any other \( \text{interm} \)-statement) to be able to deal with preclusion, in case it might occur: if \( x \) and \( y \) will justify inferring opposite statements about \( x \) — for instance, via two defaults \( (x, w, +) \) and \( (y, w, -) \) —, \( \text{interm}(x, x, y, s) \) will tell us that \( x \) should be given priority.

The second rule is the one that takes care of preclusion. Since we do it by means of the \( \text{interm} \)-statements, it is quite unlike the rules that account for preclusion in both (Sandewall, [6]) and (Simonet, [7]). Still it has the advantage of being simpler. Let us consider a concrete situation when this rule can be applied: suppose we have \( \text{isa}(a, B, +) \), \( \text{isax}(B, C, +) \), \( \text{interm}(a, D, B, +) \), and \( \text{isax}(D, C, -) \) in our set. Notice what kind of situation these statements describe. We know that it is natural to suppose that \( a \) is a \( B \) and that \( 'B\)-s are usually \( C\)-s'. However, we also know that \( a \) is a \( D \) and that \( D\)-s have priority over \( B\)-s in case both justify inference of opposite statements about \( a \), which is just what \( \text{interm}(a, D, B, +) \) tells us. Moreover, it turns out we are dealing with a case of exactly such kind, for we have \( 'D\)-s usually are not \( C\)-s' as well. Now, in order to actually prioritise \( (D, C, -) \) over \( (B, C, +) \) for \( a \), we preclude \( (B, C, +) \) for it. That is to say, we add \( \text{precl}(a, B, C, +) \) to the statement set.

The third rule accounts for adding new statements of the \( \text{isa} \)-form to the would-be extension. It even reads well (at least, when we instantiate \( s \) with either + or -): If it is natural to suppose that \( x \) is a \( y \) and there is a default \( (y, z, +) \), and neither \( (y, z, +) \) precluded, nor is it natural to suppose that \( x \) is not a \( z \), then we conclude that \( 'x \) is a \( z \)' is a natural supposition to make. Now for the other two statements that are added to \( E \) here. In light of what has been said above, addition of \( \text{interm}(x, y, z, s) \) should feel only natural. Clearly, we conclude \( \text{isa}(x, z, s) \) via \( y \) here, and want to indicate that, as long as \( x \) is concerned, \( y \) should be given priority over
The inclusion of $\neg \text{precl}(x, y, z, s)$ is a delicate matter. In short, it is our (and also Sandewall’s) way of enforcing a certain order on rule application. By adding $\neg \text{precl}(x, y, z, s)$ here, we arrange invalidation of those would-be extensions in which the rules from $R$ are instantiated in an incorrect order. That is to say, when first $\text{isa}(x, z, s)$ is derived via $(y, z, s)$, and only then $\text{precl}(z, y, z, s)$ is inferred. To a great extent, the third rule is a rule of Simonet. The only adjustment I have made is that we add $\text{interm}(x, y, z, s)$ instead of $\neg \text{isa}(x, z, -s)$.

Finally, the fourth rule is meant to add those $\text{interm}$-statements that cannot be added to the statement set by either rule (1) or rule (3), but should nevertheless be there. It is important to see in what kind of situations it is applied. The second positive condition for its instantiation — $\text{interm}(x, z, w, s)$ — will always be added to the set by means of rule (3). For the sake of simplicity, suppose that it happened in the previous step, i.e., that $\text{isa}(x, w, s)$ and $\text{interm}(x, z, w, s)$ have just been added to the set on the basis of $\text{isa}(x, z, +)$ and $\text{isax}(z, w, s)$. Notice that $\text{interm}(x, z, w, s)$ tells us that $z$ must be given priority over $w$, and that, at this point, there is nothing else we know about $w$’s relation to other nodes. However, the derivation of $\text{isa}(x, z, +)$ (and, thus, also of $\text{isa}(x, w, s)$) must have relied on other defaults — $(x, x_1, +)$, $(x_1, x_2, +)$, ..., $(x_n, z, +)$ — and in this situation $x, x_1, x_2, \ldots, x_n$ should all be given priority over $w$. Now, in fact, (4) does exactly that for us. Intuitively, what it says is this: whichever node has priority over $z$ should have priority over $w$ as well. Thus, in a sense, this rule enforces transitivity on the relation described by $\text{interm}$-statements.

The ultimate aim of any network-based account is to map any network $\Gamma$ to its corresponding conclusion set (or sets). In the path-based accounts we first construct expansions and then map them into sets of IS-A statements by means of the relation of support. Since extensions are sets of statements already, there is no need for anything akin of support. We can get the conclusion sets by simply taking subsets of extensions. That is to say, if $E$ is an extension of $\Gamma$, $\{\varphi \in E : \varphi = \text{isa}(x, y, s)\}$ must be its conclusion set.

After all the constituent parts of the rule-based system have been presented, we proceed to show that it gives the same results as the credulous (path-based) approach presented above. That is to say, we will show that any extension $E$ of a given net $\Gamma$ has a corresponding credulous expansion $\Phi$ of $\Gamma$, and vice versa. An extension $E$ is said to corresponds to an expansion $\Phi$ if the conclusion set acquired from $E$ is exactly the set of IS-A statements supported by $\Phi$. The first direction — extensions to expansions — will emerge as a combination of the following two lemmata.

**Lemma 2 (Ordering lemma).** To any sequence of increasing statement sets $E_0, E_1, \ldots$ obtained by $R$ corresponds another sequence $E'_0, E'_1$ which (i) has the same limit and (ii) complies with the following decreasing order of

---

\[ \text{One at the very least.} \]
priority on the choice of instantiation of rules from $R$: (1) the first rule; (2) the fourth rule; (3) the second rule; (4) the third rule.

Proof. See the appendix.

Lemma 3. Any consistent statement set $E$ that is obtained by the set $R$ of inference rules from a consistent inheritance net $\Gamma$ (with priority ordering as specified in the Ordering lemma) has a corresponding consistent path set $\Phi$ such that:

(i) for any $\pi \in \Phi$, $\langle \Gamma, \Phi \rangle \vDash \pi$, and

(ii) for any isa-statement, $\text{isa}(x, z, s) \in E$ iff for some path $\sigma = (x, \sigma', z, s)$, $\sigma \in \Phi$.

Proof. See the appendix.

Now comes the theorem:

Theorem 1. To any extension $E$ of a consistent inheritance network $\Gamma$ corresponds a credulous expansion $\Phi$ of $\Gamma$ such that, for all isa-statement, we have: $\text{isa}(x, z, s) \in E$ iff for some path $\sigma = (x, \sigma', z, s)$, $\sigma \in \Phi$.

Proof. Let $E$ be an extension of some arbitrary consistent inheritance net $\Gamma$. By the Ordering lemma, $E$ can be acquired respecting a certain order of priority on rule instantiation and, by lemma 3, it has a corresponding path set $\Phi$. It is easy to see that this $\Phi$ either already is an expansion or lacks only superfluous paths, and, thus, can be extended into an expansion (recall lemma 1).

Our second theorem shows that the other direction — expansions to extensions — holds just as well.

Theorem 2. For any expansion $\Phi$ of a consistent network $\Gamma$ there exists a corresponding extension $E$ such that, for any path $\pi = (x, \pi', z, s)$, $\pi \in \Phi$ iff $\text{isa}(x, z, s) \in E$.

Proof. See the appendix.

To state this in other words, we have successfully transferred the credulous approach to a different framework. One immediately noticeable improvement is that the extensions of networks that contain cycles are always finite. $\Gamma_3$ still poses a problem, but in the next section we extend the rule-based approach in order to solve it. Before proceeding to it, however, we state two nice results characterising this newly developed system.\footnote{In [5] Erik Sandewall proved some general results for systems of non-monotonic inference rules. They apply directly to our rule-based framework.}
Proposition 2. Let $E$ and $E'$ be two extensions of a network $\Gamma$, for which $E \subseteq E'$. Then $E = E'$

Proposition 3. Every union of distinct extensions of the same network $\Gamma$ is inconsistent.

4 Extension of the System: Abnormality Statements

Let us return to $\Gamma_3$. Recall that on the credulous (path-based) approach it had two equally important expansions: the intuitively correct one supported the inference of ‘it is natural to suppose that Alice has a cold’, while the other allowed for inference of the opposite statement. Let us, however, have a closer look at what makes these two ‘lines of reasoning’ differ on the intuitive level. In case the reasoner chooses to infer $isa(a, Co, +)$, it also has to conclude that Alice is an exception to the default rule ‘People with normal body temperature usually do not have a cold’. That is to say, that her temperature is normal, but she still has a cold. If, on the other hand, the reasoner chooses the other ‘line of reasoning’, it has to infer that Alice is an exception to a whole bunch of defaults. The intuition here is, of course, that the less exceptions we have, the better. In fact, the same intuition seems to lie at the very heart of John McCarthy’s circumscription formalism with its distinguished $ab$ predicate and the idea of minimising (=circumscribing) abnormalities (see (McCarthy, [4])). We will now formalise this intuition and add it to our rule-based approach. Following McCarthy, we will say that an object is $abnormal$ with respect to a certain default rule, when it forms an exception to this rule.

Our first step is to enrich the language. Since we want to be able to compare distinct extensions of the same net by the number of abnormalities each implies, it is only natural to add statements expressing abnormality. We shall write them as $abnorm(x, y, z, s)$. The intended meaning should be obvious: ‘$x$ is abnormal with respect to $(y, z, s)$’.

Now, when we have the $abnorm$-statements at our disposal, we can reformulate our non-monotonic inference rules. Our changes are of two kinds: (i) we make a slight adjustment to one of our original rules, and (ii) we add a new inference rule. Here is how a modified version of rule (2) looks like:

2. If $isa(x, y, +)$, $isax(y, z, s)$, $interm(x, v, y, +)$, and $isax(v, z, -s)$ are in $E$,

then add $precl(x, y, z, s)$ and $abnorm(x, y, z, s)$ to $E$.

Thus, the single change we make is that now, after it is applied, we do not add a $precl$-statement only, but also one of the $abnorm$-form. The motivation here should be apparent: if we conclude that a certain default is
precluded for some object, this object has to be abnormal with respect to it. We leave rules (1), (3), and (4) unchanged. A minute reflection should convince the reader that there was nothing to change about them. For the first simply translates the isax-statements into those of the isa-form, rule (3) derives new information, and the sole concern of (4) are the intermediaries.

This is all with respect to the existing rules. We still need a new rule, if we want the extensions to contain all the abnorm-formulae they should. The second rule adds an abnorm-statements in cases of preclusion, but those that have a Nixon Diamond-like shape still have to be taken care of. Our fifth rule serves exactly this purpose.

5. If isa(x, y, +), isax(y, z, s), and isa(x, z, −s) are in E, and precl(x, y, z, s) is not in E, then add abnorm(x, y, z, s) and ¬precl(x, y, z, s).

Notice that the conditions for instantiating this rule are nothing but a description of a Nixon Diamond-like situation. Let us suppose that s stands for + for the sake of simplicity. Now, we have a statement saying that it is natural to suppose that x is a y and a default (y, z, s) which is known not to be precluded for x. However, there also is another statement specifying that, in fact, it is natural to suppose that x is not a z. Clearly, in such a situation x must be abnormal with respect to (y, z, s). We add ¬precl(x, y, z, s) as well in order to invalidate those would-be extensions where this new rule is instantiated before a statement of the precl-form is derived by rule (2). That is to say, in those cases where preclusion is at issue.

Let us refer to the enhanced set of rules as R'. It is easy to show that nothing has been in the process expanding the system, i.e., that R' is at least as powerful as R.

**Proposition 4.** Any extension E acquired by the rule set R from an inheritance network Γ has a corresponding extension E' that is obtained by R' such that for any formula ϕ: if ϕ ∈ E, ϕ ∈ E'.

**Proof.** A straightforward induction on the construction of E. □

Recall our aim: it is to devise a system that is able to distinguish between different extensions by means of the number of abnormalities they imply. Now, the rule set R' works in such a way that every extension already contains the statements expressing abnormalities, and it only remains to single out the extension that minimises abnormality, i.e., contains as few abnorm-statements as possible. To this end we first define a function for each extension E:

$$\#\text{abnorm}(E) = \left| \{ \varphi \in E : \varphi = \text{abnorm}(x, y, z, s) \} \right|$$
The output of \( \#abnorm \) is the number of abnormality statements a given extension contains. In fact, that is all we need in order to define the notion of a minimal extension — an extension that actually minimises abnormalities:

**Definition 10** (Minimal extension). The minimal extension of a network \( \Gamma \) is an extension \( E \) of \( \Gamma \) such that, for no extension \( E' \) of \( \Gamma \), we have \( \#abnorm(E') < \#abnorm(E) \).

Notice that, on this definition, any network will have a minimal extension, unless, of course, it has no extensions at all. Besides, some networks will have more than one. The Nixon Diamond may serve as an example, for both of its extensions qualify as minimal.

Let us, however, return to \( \Gamma_3 \). On the present approach, it has two extensions: \( E \) and \( E' \). If we leave out the formulae serving technical purposes — negations and the *interm*-statements — and also the trivial ones — i.e., \( isax(x,z,s) \) and \( isa(x,z,s) \) such that \((x,z,s)\) is a link of \( \Gamma_3 \) —, the two extensions look as follows:

\[
E = \{ isa(a,Co,-), abnorm(a,SoThr,Co,+), abnorm(a,RuN,Co,+),
    abnorm(a,Cough,Co,+), abnorm(a,NasCon,Co,+); \}
\]

\[
E' = \{ isa(a,Co,+), abnorm(a,NormTemp,Co,-) \}.
\]

Clearly, \( \#abnorm(E) = 4 \) and \( \#abnorm(E') = 1 \). Given that \( E \) and \( E' \) are the only extensions of \( \Gamma_3 \), we see immediately that \( E' \) satisfies the condition for being its minimal extension. The two non-trivial formulae that \( E' \) contains — \( isa(a,Co,+) \) and \( abnorm(a,NormTemp,Co,-) \) — are exactly what we wanted to be able to derive on the basis of \( \Gamma_3 \). Thus, at least in this case, the minimal extension gives exactly the intuitively correct result.

### 5 Conclusion

We began with a formal presentation of a certain fairly standard path-based approach to inheritance networks. We then proceeded to develop an alternative network-based formalism that relies on a set \( R \) of four non-monotonic inference rules and proved that it maps networks into the same conclusion sets as the first one. Quite unlike its path-based counterpart, however, this new system accommodates cyclic nets and seems to allow naturally for extensions. One such was presented in the last section of the paper: we introduced statements that can say that an object is abnormal with respect to a certain default which allowed us to single out the ‘most normal’ conclusion sets, i.e., those that imply the least number of abnormalities. This was shown to suffice for dealing with such cases as that of \( \Gamma_3 \). All in all, it seems that we have successfully transferred the path-based based approach we began with to a more fruitful environment.
Appendix

Lemma 2 (Ordering lemma). To any sequence of increasing statement sets $E_0, E_1, \ldots$ obtained by $R$ corresponds another sequence $E_0, E_1'$ which (i) has the same limit and (ii) complies with the following decreasing order of priority on the choice of instantiation of rules from $R$: (1) the first rule; (2) the fourth rule; (3) the second rule; (4) the third rule.

Proof. By induction on the construction of a sequence.

Base case. Trivial.

Inductive step. We suppose that the claim holds for all sequences constructed in $n - 1$ steps and show that it holds for sequences constructed in $n$ steps as well. There are four inference rules, and, hence, four cases to consider.

(1) $E_n$ is acquired by an application of the first rule. Here $E_n = E_{n-1} \cup \Delta$, where $\Delta$ stand for $\{\text{isa}(x, y, s), \text{intern}(x, x, y, s)\}$. Now, by the inductive hypothesis, we know that $E_0, E_1, \ldots, E_{n-1}$ has a corresponding sequence $E_0, E_1', \ldots, E_{n-1}'$ that obeys the priority ordering. We set $E_0, E_0 \cup \Delta, E_1' \cup \Delta, \ldots, E_{n-1}' \cup \Delta$ to be the sequence corresponding to $E_0, E_1, \ldots, E_{n-1}, E_n$.

(2) $E_n$ is obtained by an instantiation of rule (2). Here the sequence we are dealing with looks as follows: $E_0, E_1, \ldots, E_{n-1}, E_{n-1} \cup \Delta$, where $\Delta = \{\text{pred}(x, y, z, s)\}$. By the inductive hypothesis, it has an order-obeying counterpart. Now, let $E_i'$ be the first statement set in it that contains all the conditions necessary for inferring $\text{precl}(x, y, z, s)$ and $E_j'$ ($i \leq j \leq n - 1$) — the first statement set after $E_j'$ that is acquired by rules other than (1) or (4). Thus, the sequence at issue looks as follows: $E_0, E_1', \ldots, E_i', E_j', E_{j-1}', \ldots, E_{n-1}$.

We set $E_0, E_1, \ldots, E_{n}$ to correspond to: $E_0, E_1', \ldots, E_i', E_j', E_{j-1}' \cup \Delta, E_j' \cup \Delta, \ldots, E_{n-1} \cup \Delta$.

(3) $E_n$ is obtained by applying rule (3). Here our sequence is $E_0, E_1, \ldots, E_{n-1}, E_{n-1} \cup \{\text{isa}(x, z, s), \neg\text{precl}(x, y, z, s), \text{intern}(x, y, z, s)\}$. This is the easy case. If $E_0, E_1', \ldots, E_{n-1}$ is the sequence corresponding to $E_0, E_1, \ldots, E_{n-1}$, let $E_0, E_1', \ldots, E_{n-1}'$, $E_{n-1}' \cup \{\text{isa}(x, z, s), \neg\text{precl}(x, y, z, s), \text{intern}(x, y, z, s)\}$ be the one corresponding to $E_0, E_1, \ldots, E_n$.

(4) $E_n$ is a result of an instantiation of the fourth rule. Here the sequence is $E_0, E_1, \ldots, E_{n-1}, E_{n-1} \cup \Delta$, where $\Delta = \{\text{intern}(x, y, w, s)\}$. $E_0, E_1, \ldots, E_{n-1}$ has a corresponding sequence that complies with the above order. Let $E_i'$ be the first statement set that contains the formulae that
are needed to infer \( \text{interm}(x, y, w, s) \) and let \( E'_j \) (with \( i \leq j \leq n - 1 \)) be the first set after \( E'_i \) that is not acquired by rule (1). Thus, the sequence we are dealing with here must look as follows: \( E_0, E'_1, \ldots, E'_i, \ldots, E'_{j-1}, E'_j, \ldots, E_{n-1} \). We set \( E_0, E_1, \ldots, E_n \) to correspond to:

\[
E_0, E'_1, \ldots, E'_i, \ldots, E'_{j-1}, E'_j, \ldots, E_{n-1} \cup \Delta, E'_j \cup \Delta, \ldots, E_{n-1} \cup \Delta.
\]

Lemma 3. Any consistent statement set \( E \) that is obtained by the set \( R \) of inference rules from a consistent inheritance net \( \Gamma \) (with priority ordering as specified in the Ordering lemma) has a corresponding consistent path set \( \Phi \) such that:

(i) for any \( \pi \in \Phi \), \( \langle \Gamma, \Phi \rangle \models \pi \), and

(ii) for any \( \text{isa} \)-statement, \( \text{isa}(x, z, s) \in E \) iff for some path \( \sigma = (x, \sigma', z, s) \), \( \sigma \in \Phi \).

Proof. The proof is by induction on the construction of \( E \). Our order of priority on the choice of rule application is as follows (the reason we need it will become clear as the proof proceeds): (1) The first rule; (2) the fourth rule; (3) the second rule; (4) the third rule.

Base case. All formulae of \( E_0 \) are of the \( \text{isax} \)-form. Now, since there are no formulae of the \( \text{isa} \)-form in \( E_0 \), we can let \( \emptyset \) be the set corresponding to \( E_0 \).

Inductive step. We suppose that the claim holds for all statement sets that have been constructed in \( n - 1 \) steps and show that it hold for the \( n \)-th step as well. Since there are four inference rules in our system, we have to consider four cases.

1. \( E_n \) is acquired by an application of the first rule. Then \( \text{isax}(x, z, s) \in E_{n-1} \) and \( E_n = E_{n-1} \cup \{ \text{isa}(x, z, s), \text{interm}(x, y, s) \} \). Clearly, we have \( (x, z, s) \in \Gamma \). Now, by the inductive hypothesis, we know that there is a path set \( \Phi' \) such that \( \langle \Gamma, \Phi' \rangle \models \pi \) for any \( \pi \in \Phi' \) and that it contains all and only the paths corresponding to the \( \text{isa} \)-formulae of \( E_{n-1} \). Since \( (x, z, s) \) is a link of \( \Gamma \), we can be sure that \( \langle \Gamma, \Phi' \rangle \models (x, z, s) \). We set \( \Phi' \cup \{(x, z, s)\} \) to be the path set corresponding to \( E_n \).

2. \( E_n \) is acquired by an application of the second inference rule. Since \( E_n \) does not contain any additional \( \text{isa} \)-formulae but those already present in \( E_{n-1} \), in this case the claim holds by the inductive hypothesis alone.

3. \( E_n \) is a result of applying the third inference rule. Here \( E_n = E_{n-1} \cup \{ \text{isa}(x, z, s), \neg \text{precl}(x, y, z, s), \text{interm}(x, y, z, s) \} \). We can be sure that \( \text{isa}(x, y, +) \) and \( \text{isax}(y, z, s) \in E_{n-1} \) and also that \( \text{precl}(x, y, z, s) \) and \( \text{isa}(x, z, -s) \notin E_{n-1} \) (negative conditions of applying the 3rd rule).
By the inductive hypothesis we know that there is a path set $\Phi'$ corresponding to $E_{n-1}$. Since $isa(x, y, +) \in E_{n-1}$, $(x, \sigma, y, +) \in \Phi'$. Since $isax(y, z, s) \in E_{n-1}$, $\Gamma$ must contain a link of the form $(y, z, s)$. This suffices to conclude that $\sigma = (x, \sigma', y, z, s)$ is constructible in $(\Gamma, \Phi')$. Further, since $isa(x, z, -s) \notin E_{n-1}$, we can be sure that $\sigma$ is not conflicted in $(\Gamma, \Phi')$. Now, it remains to show the following:

**Claim 1.** $(x, \sigma', y, z, s)$ is not precluded in $(\Gamma, \Phi')$.

**Proof of claim.** Suppose to the contrary, i.e., that there is some node $v'$ such that (i) $v'$ is an intermediary to the path $\sigma$ in the path set $\Phi'$ and (ii) $\Gamma$ contains a direct link of the form $(v', z, s)$. (i) tells us that $isax(v', z, s) \in E_{n-1}$. (ii) tells us that either $v'$ is a node of $\sigma$ itself or $\Phi'$ must contain a path of the form $(x, \rho, v', \rho', w, y, +)$. If the former is the case, $interm(x, v', y, +)$ must have been added at some point in the process of constructing $E_{n-1}$, i.e., either simultaneously or after the addition of $isa(x, y, +)$. Otherwise, $E_n$ could not have been acquired by the application of the third rule (recall our priority ordering on rule instantiation). If the latter is the case, at some point in the construction of our statement set the third rule must have been applied on $isa(x, w, +)$ and $isax(w, y, +)$. Now, given that all the other rules must have been applied before the acquisition of $E_n$ (again, constraints on constructions), $interm(x, v', y, +)$ must already be in $E_{n-1}$. In either case we get a contradiction, for the fact that we have $precl(x, y, z, s) \notin E_{n-1}$ tells us that for no $v$ we have both: $interm(x, v, y, +)$ and $isax(v, z, -s) \in E_{n-1}$ (again, the constraints on rule application).

Now, since $(x, \sigma', y, z, s)$ is (a) constructible, (b) not conflicted, and (c) not precluded in $(\Gamma, \Phi')$, we can be sure that $(\Gamma, \Phi') \sim \sigma$. So we set $\Phi' \cup \{\sigma\}$ to be the set corresponding to $E_n$.

The reader is welcome to check that we still have $(\Gamma, \Phi') \sim \pi$, for all $\pi \in \Phi$.

(4) Same as (2).

This concludes the proof.

**Theorem 2.** For any expansion $\Phi$ of a consistent network $\Gamma$ there exists a corresponding extension $E$ such that, for any path $\pi = (x, \pi', z, s)$, $\pi \in \Phi$ iff $isa(x, z, s) \in E$.

**Proof.** We need to define the notion of a path length for this proof. Let the length of a path $\pi$, $len(\pi)$, be the number of links it consists of. Thus, for $\pi' = (a, B, C, D, +)$, we have $len(\pi') = 3$. Notice also that the number of nodes of any given path $\pi$ is $len(\pi) + 1$. Now we turn to the proof.
Let \( \Phi \) be an expansion of an arbitrary inheritance network \( \Gamma \). Let \( n \) be the length of the longest path in \( \Phi \). Now for each \( i, 1 \leq i \leq n \), we take the corresponding subset \( \Phi_i \) of \( \Phi \): \( \Phi_i = \{ \pi \in \Phi : \text{len}(\pi) = i \} \).

These subsets form a sequence \( \Phi_1, \Phi_2, \ldots, \Phi_n \) such that \( \bigcup_{i=1}^n \Phi_i = \Phi \). Notice that for any path \( \pi \) of length \( i \) such that \( 1 < i \leq n \) we have the following: \( \pi \) is of the form \( (x, \pi', y, z, s) \), \( (x, \pi', y, +) \in \Phi_{i-1} \), and \( (y, z, s) \in \Phi_1 \). Otherwise, \( \pi \) would not be constructible and, thus, \( \Phi \) could not have been an expansion. More generally, for any \( i, 1 < i \leq n \), and any \( \pi \in \Phi_i \), we have \( \langle \Gamma, \Phi_{i-1} \rangle \models \pi \).

Now we will construct a sequence of increasing sets of statements \( E_0, E_1, E_2, \ldots \), making use of our sequence \( \Phi_1, \Phi_2, \ldots \).

- Set \( E_0 = \{ \text{isax}(x, z, s) : (x, z, s) \in \Gamma \} \).

- Let \( k = |\Phi_1| \). \( E_1, \ldots, E_k \) are constructed from \( \Phi_1 \). Let each \( E_j \), where \( 1 \leq j \leq k \), be constructed as follows: for each \( \pi = (x, z, s) \in \Phi_1 \), apply the 1st inference rule on the corresponding \( \text{isax}(x, z, s) \in E_{j-1} \). Notice that \( \bigcup_{j=1}^k E_j = \{ \text{isa}(x, z, s) : (x, z, s) \in \Gamma \} \cup \{ \text{interm}(x, x, z, s) : (x, z, s) \in \Gamma \} \).

- Now we specify what to do with each \( \Phi_i \) such that \( 1 < i \leq n \). Let \( k = |\Phi_i| \). For each \( \pi_j = (x, \pi', y, z, s) \in \Phi_i \), where \( 1 \leq j \leq k \), we do the following.

  (1) Let \( E_{f-1} \) be the last statement set we have constructed. Apply inference rule (3) on the corresponding \( \text{isa}(x, y, +) \) and \( \text{isax}(y, z, s) \in E_{f-1} \), thus acquiring \( E_f \). It is easy to verify that the rule can be applied here.

  (2) Construct the next statements sets \( E_{g+1}, \ldots, E_g \) by continuous applications of the fourth inference rules, until it is not possible to apply it anymore. Notice that here we have \( f + 1 \leq g \).

  (3) In a similar fashion, construct the statement sets \( E_{g+1}, \ldots, E_h \) (where \( g + 1 \leq h \) by instantiating the second rule of inference).

Stop when no further applications are possible.

After all the path \( \pi_j \) have been considered, proceed to \( \Phi_{i+1} \).

Now, it is obvious that each \( E_i, 1 < i \) of the sequence \( E_0, E_1, E_2, \ldots \) is acquired by an application of one of the inference rules from \( R \), and it is not difficult to check that its limit \( E \) is an extension of \( \Gamma \). For the reasons of space, I leave this to the reader.
References


