

Modeling Gauss Markov Random Fields at Multiple Resolutions¹

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Abstract — A multiresolution model for Gauss Markov random fields (GMRF) is presented. Based information theoretic measure, a technique is presented to estimate the GMRF parameters of the process at lower resolution given the parameters at the fine resolution.

There has been an increasing interest in using statistical techniques for modeling and processing images in the computer vision community over the last decade. Most of the research has been restricted to Markov random field (MRF) models, rightly so, because of the local statistical dependence of images. MRF models have been used to characterize prior beliefs about continuity of image features such as textures, edges, region labels etc. Also, establishment of the connection between MRF models and Gibbs distribution has resulted in an influx of many statistical mechanics techniques like simulated annealing, mean field methods in image processing. Introduction of the Markov models in a Bayesian formalism has resulted in a unified, coherent framework for processing images which enables posing many image processing problems as statistical inference problems. MRF models have been used by various researchers [3], [5] in image modeling. The main drawback of the MRF techniques is that the optimization schemes associated with the energy functions are iterative (not recursive) and are usually computationally expensive. One way to reduce the computational burden is to use multiresolution techniques [6], [7], [2]. In this paper, we present a multiresolution model for Gauss Markov random fields (GMRF), where the resolution transformation is obtained by subsampling with or without local averaging.

Let $\Omega^{(\cdot)} = \{(i, j) : 0 \leq i \leq M - 1, 0 \leq j \leq M - 1\}$ be a lattice on which a GMRF is defined. The superscript stands for the level in the image pyramid, $\Omega^{(\cdot)}$ being the lattice at the fine resolution, $\Omega^{(\cdot)}$ represents the lattice which is obtained by subsampling $\Omega^{(\cdot)}$, k times. Let $X^{(k)}$ represent a random vector, obtained by ordering the random variables on the two-dimensional lattice $\Omega^{(\cdot)}$, through a row-wise scan. The elements of $\Omega^{(\cdot)}$ are indexed by s, p, q and t , where $s = (s_1, s_2)$. Let

$X^{(0)}$ be modeled by a GMRF, then the joint probability density function of $X^{(0)}$ can be written as follows:

$$P^{(0)}(X^{(0)} = x) = \frac{1}{(2\pi)^{\frac{MN}{2}} (\det \Sigma^{(0)})^{\frac{1}{2}}} \exp\left\{-\frac{1}{2} x^T [\Sigma^{(0)}]^{-1} x\right\} \quad (1)$$

where $\Sigma^{(0)}$ is the covariance matrix of $X^{(0)}$.

Equivalently, the process $X^{(0)}$ can be written in terms of a non-causal autoregressive representation. For a site s , let $\eta^{(0)}$ be the asymmetric neighborhood, (the complete symmetric neighborhood can be obtained by $(r : r \in \eta^{(0)}) \cup (-r : r \in \eta^{(0)})$) which contains the set of sites that are chosen to be the neighbors of $X_s^{(0)}$. We always use r to index into the neighbor set in the rest of the paper.

$$X_s^{(0)} = \sum_{r \in \eta^{(0)}} \theta_r^{(0)} (X_{s+r}^{(0)} + X_{s-r}^{(0)}) + e_s^{(0)}$$

where $e_s^{(0)}$, is zero mean, Gaussian noise, with autocorrelation given by:

$$E\{e_s^{(0)} e_{s+r}^{(0)}\} = \begin{cases} [\sigma^{(0)}]^2 & \text{if } r = 0 \\ -\theta_r [\sigma^{(0)}]^2 & \text{if } r \in \eta^{(0)} \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

Hence the GMRF process can be completely characterized by the set of parameters $\{\theta_r^{(0)}, [\sigma^{(0)}]^2\}$. It can be shown that the GMRFs lose Markovianity on subsampling resolution transformation. However, if the lower resolution data are modeled by the exact non-Markov Gaussian processes, conventional optimization techniques based on Markov properties cannot be applied. We present a *local conditional distribution invariance* Markov approximation at lower resolutions such that, if $P^{(k)}(X^{(k)})$ is the non-Markov pdf at the k th resolution then a GMRF approximation $P_g^{(k)}(X^{(k)})$ can be obtained such that,

$$P_g^{(k)}(X_s^{(k)} / X_{s+r}^{(k)}, r \in \eta^{(k)}) = P^{(k)}(X_s^{(k)} / X_{s+r}^{(k)}, r \in \eta^{(k)}). \quad (3)$$

The invariance of local conditional distribution is a desirable property especially because most of the optimization techniques such the simulated annealing [4], and iterated conditional mode [1] use the local conditional

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distribution. The parameters $(\underline{\theta}^{(k)}, [\sigma^{(k)}]^2)$ corresponding to $P_g^{(k)}(X^{(k)})$ are obtained as follows:

$$\underline{\theta}^{(k)} = \underset{\underline{\alpha}}{\text{arg min}} E_{P^{(k)}}[X_s^{(k)} - \sum_{r \in \eta^{(k)}} \alpha_r (X_{s+r}^{(k)} + X_{s-r}^{(k)})]^2 \quad (4)$$

and using the $\underline{\theta}^{(k)}$ obtained, we can estimate the $[\sigma^{(k)}]^2$ as,

$$[\sigma^{(k)}]^2 = E_{P^{(k)}}[X_s^{(k)} - \sum_{r \in \eta^{(k)}} \theta_r^{(k)} (X_{s+r}^{(k)} + X_{s-r}^{(k)})]^2. \quad (5)$$

The minimization requires the autocorrelation values $E_{P^{(k)}}(X_s^{(k)} X_{s+r}^{(k)})$ which can be computed, given the GMRF parameters for $X^{(0)}$ as shown below, for subsampling resolution transformation.

$$\begin{aligned} X_s^{(k)} &= X_{2^k s}^{(0)} \\ E_{P^{(k)}}(X_s^{(1)} X_{s+r}^{(1)}) &= E_{P^{(0)}}(X_{2^k s}^{(0)} X_{2^k(s+r)}^{(0)}) \\ E_{P^{(0)}}(X_p^{(0)} X_q^{(0)}) &= \frac{1}{MN} \sum_{s \in \Omega^{(0)}} \frac{(\lambda_{s_1}^{p_1} \lambda_{s_2}^{p_2})(\lambda_{s_1}^{q_1} \lambda_{s_2}^{q_2})}{1 - 2[\underline{\theta}^{(0)}]^T \phi_s} \end{aligned}$$

where $\lambda_i = \exp(\sqrt{-1} \frac{2\pi i}{M})$.

The GMRF approximation presented above provides local conditional distribution invariance. We also show that, $P_g^{(k)}(X^{(k)})$ is “close” to $P^{(k)}(X^{(k)})$ by comparing the power spectral density functions. Let $S_\omega^{(k)}$ and $GS_\omega^{(k)}(m)$ be the power spectral density functions associated with $P^{(k)}(X^{(k)})$ and $P_g^{(k)}(X^{(k)})$ respectively. It can be shown that the normalized absolute difference $D(k, m) = \frac{\sum_{(\omega \in \Omega^{(k)})} |S_\omega^{(k)} - GS_\omega^{(k)}(m)|}{\text{size}(\Omega^{(k)})}$, where $\text{size}(\Omega^{(k)})$ is the number of lattice sites in $\Omega^{(k)}$, is small. In addition, different sets of parameters at the fine resolution can result in the same statistics after subsampling. It can be shown that for the first order case, the sets of parameters $(\theta_{(1,0)}, \theta_{(0,1)})$, $(-\theta_{(1,0)}, \theta_{(0,1)})$, $(\theta_{(1,0)}, -\theta_{(0,1)})$, $(-\theta_{(1,0)}, -\theta_{(0,1)})$ result in the same statistics after one level of subsampling and similarly the sets of parameters $(\theta_{(1,0)}, \theta_{(0,1)}, \theta_{(1,1)}, \theta_{(-1,1)})$, $(-\theta_{(1,0)}, \theta_{(0,1)}, -\theta_{(1,1)}, -\theta_{(-1,1)})$, $(\theta_{(1,0)}, -\theta_{(0,1)}, -\theta_{(1,1)}, -\theta_{(-1,1)})$, $(-\theta_{(1,0)}, -\theta_{(0,1)}, \theta_{(1,1)}, \theta_{(-1,1)})$ for the second order case. It is possible to obtain results for higher orders and higher levels of subsampling similarly. We have applied this multiresolution model for texture segmentation at multiple resolutions. In general this can be used for several applications that use GMRF models.

References

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