Diffeomorphic Active Contours

Felipe Arrate

Applied Mathematics and Statistics
Center of Imaging Science
Johns Hopkins University

April 2009
Outline

1. Introduction

2. The Framework
   - Segmentation Function
   - Gradient Descent
   - Projected Gradient Descent Flow

3. Numerical Implementation
   - AKM
   - Re-gridding

4. Applications to Segmentation
   - 2D segmentation
   - 3D segmentation

5. Future Work
Space of Anatomies

- Anatomical structures \( \cong \text{Templates} \).

- Shapes, or “objects than can be deformed”, are considered as points on a manifold.

- Shape deformation is defined as the action of a group of diffeomorphisms \( \text{Diff} \).

![Diagram of brain structures with labels for amygdala, caudate, hippocampus, pallidus, putamen, thalamus, ventricle]
The setting shares similarities with the mechanics of perfect fluids. As emphasized by Arnold (1966), many fundamental properties of Lie groups in rigid body mechanics can be formally extended from the finite to the infinite dimensional setting, allowing the study of shapes via diffeomorphisms acting on them.

This approach allows to focus the modeling effort on the diffeomorphism group $\text{Diff}$, rather than the family of objects being deformed.
Space of 'Friends'

From rotation on $SO(3)$

$SO(3)$
Space of 'Friends'

To the action of elements of $\text{Diff}$. 
Space of 'Friends'

To the action of elements of $Diff$.
Space of 'Friends'

To the action of elements of $\text{Diff}$. 
Space of 'Friends'

It has been a long journey:
It has been a long journey:
Space of 'Friends'

It has been a long journey:
Space of 'Friends'

It has been a long journey:

In the following an application to image segmentation will be shown.
As usual, large diffeomorphisms will be consider as the flow associated to the following non-autonomous differential equation

\[
\frac{\partial \varphi_t(\cdot)}{\partial t} = v_t \circ \varphi_t(\cdot) \quad \varphi_0^v = id
\] (1)
As usual, large diffeomorphisms will be consider as the flow associated to the following non-autonomous differential equation

\[
\frac{\partial \varphi_t(\cdot)}{\partial t} = v_t \circ \varphi_t(\cdot) \quad \varphi_0^v = id
\]  

For a fix \( t \), \( v_t \in \mathcal{V} \) will be assumed canonically embedded in \( C^1_0(\Omega, \mathbb{R}^d) \) and bounded under its norm. Then \( \mathcal{V} \) will generate a group of diffeomorphisms that we will denote as

\[ \mathcal{G}_\mathcal{V} = \{ \varphi_t^v, v_t \in \mathcal{V}, \text{ for each } t \} \]
As usual, large diffeomorphisms will be consider as the flow associated to the following non-autonomous differential equation

\[
\frac{\partial \varphi_t(\cdot)}{\partial t} = v_t \circ \varphi_t(\cdot) \quad \varphi_0 = id
\]  \hspace{1cm} (1)

For a fix \( t \), \( v_t \in \mathcal{V} \) will be assumed canonically embedded in \( C^1_0(\Omega, \mathbb{R}^d) \) and bounded under its norm. Then \( \mathcal{V} \) will generate a group of diffeomorphisms that we will denote as

\[
\mathcal{G}_\mathcal{V} = \{ \varphi^v_t, v_t \in \mathcal{V}, \text{ for each } t \}
\]

Given a positive definite kernel \( K : \Omega \times \Omega \mapsto \mathbb{R} \) we construct a unique Hilbert space \( \mathcal{V} \) (up to isometries) (RKHS) as the completion of the space of linear combinations of the type

\[
v(\cdot) = \sum_i K(\cdot, x_i) \alpha_i \quad \alpha_i \in \mathbb{R}^d
\]  \hspace{1cm} (2)
Segmentation Function

Assume that a scalar function $f_n : \Omega_2 \rightarrow \mathbb{R}$, $x \mapsto \mathbb{R}$ that measures the likelihood of a point at $p \in \Omega_2$ with tissue $l_n$ to belong to a similar tissue defined on $q = \varphi^{-1}(p) \in \Omega_1$ is given.

We propose the following energy term

$$J(\varphi_t) = \sum_{n=0}^{M} \int_{\Omega_1^n} f_n(\varphi_t(q)) |D\varphi_t(q)| \, dq$$

or

$$J(\varphi_t) = \sum_{n=1}^{M} \int_{\Omega_1^n} \mathcal{H}_n(\varphi_t(q)) |D\varphi_t(q)| \, dq$$

where $\mathcal{H}_n(\cdot) = \left( f_n(\cdot) - f_0(\cdot) \right)$

This energy can be seen as a measure of how well the shapes inside $\Omega_1$ are deformed to match the tissues in $\Omega_2$. 
The Framework

Segmentation Function

$\Omega_1$  
Template

$\Omega_2$  
Target

$\varphi(q_1)$

$\varphi(q_2)$

$\varphi(\Omega^1_1)$

Figure: Energy calculation
Shape Gradient

- Write the action of the shape derivative in the direction of a perturbation as the linear form $dJ_\varphi(u_t)$
- Assume a right invariant Riemannian metric $\langle \cdot, \cdot \rangle_{T_\varphi \mathcal{G}_V}$ defined on every $\varphi \in \mathcal{G}_V$ such that
  \[\|T_{id}R_\varphi v_t\|_{T_\varphi \mathcal{G}_V} = \|v_t\|_V, \forall \varphi \in \mathcal{G}_V\]
- The direction of steepest descent with respect to the shape function $J(\Omega)$ is given by the unique vector $\overline{\nabla} J_\varphi \in T_{id} \mathcal{G}_V \cong V$.

For any $v_t \in \mathcal{V}$

\[
\left. \frac{d}{d\varepsilon} J(\varphi + \varepsilon(v_t \circ \varphi)) \right|_{\varepsilon=0} = (dJ_\varphi | v_t \circ \varphi) = \langle \overline{\partial} J_\varphi_t | v_t \rangle = \langle \overline{\nabla} J_\varphi, v_t \rangle_V
\]
Projected Gradient Descent Flow

Assume a set of \textit{control points} $x_1, x_2, \ldots, x_N$ defined on the boundary of every closed surface (tissue) on $\Omega_1$. Given the RKHS $\mathcal{V}$, take

$$\mathcal{W} = \left\{ \sum_{i=1}^{N} K_0(\cdot, \varphi_t(x_i)) \alpha_i(t); \alpha_1(t), \ldots, \alpha_N(t) \in \mathbb{R}^d \right\} \subset \mathcal{V}$$

$K_0(\cdot, x)$ is a kernel selected to ensure the inclusion of $\mathcal{W}$ in $\mathcal{V}$.

Define gradient $\tilde{\nabla}J_{\varphi_t}$ to be the projection of $\nabla J_{\varphi_t}$ on $\mathcal{W}$ as

$$\tilde{\nabla}J_{\varphi_t}(\cdot) = \sum_{i=1}^{N} K_0(\cdot, \varphi_t(x_i)) \alpha_i(t)$$
Projected Gradient Descent Flow

Assume a set of *control points* \( x_1, x_2, \ldots, x_N \) defined on the boundary of every closed surface (tissue) on \( \Omega_1 \). Given the RKHS \( \mathcal{V} \), take

\[
\mathcal{W} = \left\{ \sum_{i=1}^{N} K_0(\cdot, \varphi_t(x_i)) \alpha_i(t); \ \alpha_1(t), \ldots, \alpha_N(t) \in \mathbb{R}^d \right\} \subset \mathcal{V}
\]

\( K_0(\cdot, x) \) is a kernel selected to ensure the inclusion of \( \mathcal{W} \) in \( \mathcal{V} \).

Define gradient \( \widetilde{\nabla} J_{\varphi_t} \) to be the projection of \( \nabla J_{\varphi_t} \) on \( \mathcal{W} \) as

\[
\widetilde{\nabla} J_{\varphi_t}(\cdot) = \sum_{i=1}^{N} K_0(\cdot, \varphi_t(x_i)) \alpha_i(t)
\]

The orthogonality projection constraint will require that for all \( j \)

\[
\left\langle \nabla J_{\varphi_t} - \widetilde{\nabla} J_{\varphi_t}, \sum_{i=1}^{N} K_0(\cdot, \varphi_t(x_i)) a_j \right\rangle_{\mathcal{V}} = 0
\]

\[
\left( \partial J_{\varphi_t} \mid \sum_{i=1}^{N} K_0(\cdot, \varphi_t(x_i)) a_j \right) = \left\langle \sum_{i=1}^{N} K_0(\cdot, \varphi_t(x_i)) \alpha(t), \sum_{i=1}^{N} K_0(\cdot, \varphi_t(x_i)) a_j \right\rangle_{\mathcal{V}}
\]
The Framework

Projected Gradient Descent Flow

\[
\begin{align*}
F_j(t) = & \left( \bar{\partial} J_{\varphi_t} \middle| \sum_{i=1}^{N} K_0(\cdot, \varphi_t(x_i)) a_j \right) = \\
& \sum_{i=1}^{Nd} \alpha_i(t) \left( \sum_{i=1}^{N} K_0(\cdot, \varphi_t(x_i)) a_i , \sum_{i=1}^{N} K_0(\cdot, \varphi_t(x_i)) a_j \right) \hat{V} \\
M_{ij}(t) &
\end{align*}
\]
Projected Gradient Descent Flow

\[
F_j(t) = \left( \partial J_{\varphi_t} \mid \sum_{i=1}^{N} K_0(\cdot, \varphi_t(x_i)) \alpha_j \right) = \\
\sum_{i=1}^{Nd} \alpha_i(t) \left\langle \sum_{i=1}^{N} K_0(\cdot, \varphi_t(x_i)) \alpha_i, \sum_{i=1}^{N} K_0(\cdot, \varphi_t(x_i)) \alpha_j \right\rangle \text{V}
\]

\[
F_j(t) = \sum_{i=1}^{N} \int_{\Omega_1} \left[ \nabla H^T(\varphi_t(q)) K_0(\varphi_t(q), \varphi_t(x_i)) + H(\varphi_t(q)) \nabla_1 K_0(\varphi_t(q), \varphi_t(x_i)) \right] a^i_j \cdot |D \varphi_t(q)| dq
\]

\[
F(t) = M_t \alpha(t)
\]

Solving this system of equations for all \( \alpha_k(t) \), yields \( \tilde{\nabla} J_{\varphi_t} \), and a projected gradient descent process can be performed.
Discrete Energy

The equation for the energy term

\[ J(\varphi_t) = \sum_{n=1}^{M} \int_{\Omega_1^n} \mathcal{H}_n(\varphi_t(q)) |D\varphi_t(q)| \, dq \]

can be approximated as

\[ U(\varphi_t) = \sum_{n=1}^{M} \sum_{r=1}^{R_n} \mathcal{H}_n \left( \varphi_t \left( z_r^{(n)}(0) \right) \right) V_r^{(n)}(t) \]  \hspace{1cm} (4)

where:

\[ V_r(t) = |D\varphi_t(z_r(0))| : \text{Evolving volume of each voxel.} \]
\[ z_1(t), \ldots, z_{R_1}(t) : \text{Position of the center of each voxel at time } t. \]
Alternating Kernel Mixture method (AKM)

The likelihood function $f_n(\cdot)$ will be the output of the Alternate Kernel Mixture method (AKM), whose algorithm yields the probability of each point on the target $\Omega_2$ to belong to a specific tissue.
After all these approximations, the projected gradient descent process is performed solving

\[ \frac{d x_j(t)}{dt} = \sum_{i=1}^{N} K_0(x_j(t), x_i(t)) \alpha_i(t) \quad (5) \]

\[ \frac{d z_r^{(n)}(t)}{dt} = \sum_{i=1}^{N} K_0\left(z_r^{(n)}(t), x_i(t)\right) \alpha_i(t) \quad (6) \]

\[ \frac{d V_r^{(n)}(t)}{dt} = V_r^{(n)}(t) \cdot \sum_{i=1}^{N} \nabla^T_1 K_0\left(z_r^{(n)}(t), x_i(t)\right) \alpha_i(t) \quad (7) \]
Re-gridding

a) Deformed Template, $t = T$

b) New grid

c) Original Template, $t = 0$

d) New Labeled Template, $t = T$

\[ \varphi_{T}(q) = p \]

\[ \varphi_{T}^{-1}(p) \]

Figure 2. Re-gridding Process:

- At time $t = T$ one of the volumes surpasses the limits $[V_{\text{min}}, V_{\text{max}}]$. Then, a new grid is defined over an area enclosing the tissue in $a$ and $b$, and every point $p$ is evolved back to the initial template $c$, to be labeled accordingly in a new template $d$.

- This is mainly due to the selected values for $\sigma = 15$ and $\sigma_0 = 30$.

Conclusions and Future Work

In this study we proposed a diffeomorphic active contour solution to the segmentation of medical images in which anatomies are presented as deformable templates being transformed by the action of members of a group of diffeomorphisms. The procedure minimizes a proposed segmentation energy based on a measure of the likelihood of every voxel to belong to a certain tissue, obtained using the AKM method on the original data. In particular, the definition of a finite set of control points on the boundary of the templates yields a projected gradient descent on the space $\text{Diff}$ that allowed a topologically invariant deformation of the initial structures, as well as a robust anatomical correlation between the template and the target.

Due to the common presence of weak boundaries between similar tissue but belonging to different structures, in the future we propose an improvement on the current model with a redefinition of the energy term to take into consideration prior knowledge of the level of deformation allowed, and the optimal position of the set of control points. Also as it was previously suggested, a line search method will be included in the program to speed up results and accurately stop the algorithm based in a null value for the gradient $\tilde{\nabla} U_{\phi}.$

As a future development on the model, we plan to extend these concepts to video tracking. The model that we propose for diffeomorphic video tracking starts from the result of the segmenting algorithm (time 0), and then it will compute the new position of the deformed template on the next image of the video sequence subject to a diffeomorphic constraint.
Applications to Segmentation (2D)

Segmentation of a two dimensional dumbbell. The size of the target image is $124 \times 252$ pixels. The initial is a binary disc with 30 control points regularly positioned in its boundary.
Applications to Segmentation (2D)

Heart Segmentation, 2D. Heart segmentation in 2D. The target is an MRI of size $142 \times 102$ pixels. Note the presence of a pacemaker, that corrupts the right section of the left ventricle wall. ($\sigma = 20$, $\sigma_0 = 30$)
Applications to Segmentation (3D)

Segmentation of a 3 dimensional dumbbell. Dumbbell segmentation in 3D. The target is a binary image of size $62 \times 62 \times 126$ voxels, corrupted by 20% of Gaussian noise. ($\sigma = 7, \sigma_0 = 14$).
Heart Segmentation, 3D. Heart segmentation in 3D. The target is an MRI of size $103 \times 142 \times 102$ voxels. Note the presence of a pacemaker and papillary muscle that are not segmented by the algorithm. As input values we used $\sigma = 15$ and $\sigma_0 = 30$. 
Heart Segmentation, 3D. Heart segmentation in 3D. The target is an MRI of size $103 \times 142 \times 102$ voxels. Note the presence of a pacemaker and papillary muscle that are not segmented by the algorithm. As input values we used $\sigma = 15$ and $\sigma_0 = 30$. 
Heart Segmentation, 3D. Heart segmentation in 3D. The target is an MRI of size $103 \times 142 \times 102$ voxels. Note the presence of a pacemaker and papillary muscle that are not segmented by the algorithm. As input values we used $\sigma = 15$ and $\sigma_0 = 30$.
Future Work: Boundary based segmentation

Using the divergence theorem: Alternate expression for Segmentation Energy

\[ d\tilde{J}(\varphi_t)[u \circ \varphi_t] = \int_{\partial\Omega_1} f u \cdot n \, ds \]

with \( n \) the outward normal to \( \partial\Omega_1 \), and the gradient descent flow is

\[ \frac{\partial \varphi(t, \cdot)}{\partial t} = - \int_{\partial\varphi(\Omega_1)} K(y, \varphi(t, \cdot)) f(y) \cdot n \, ds(y) \]
Future Work: Boundary based segmentation
Future Work

1. Robustness of the method to changes in:
   - Initial position
   - Number of Control Points
   - Control Points Position.

2. Linesearch to define step size.

3. Redefinition of Energy term to consider biological prior knowledge.
Thanks!