Classification: Rademacher Complexity

Machine Learning: Alvin Grissom II
University of Colorado Boulder

LEcTure

Slides adapted from Rob Schapire
Setup

Nothing new . . .

- Samples $S = ((x_1, y_1), \ldots, (x_m, y_m))$
- Labels $y_i = \{-1, +1\}$
- Hypothesis $h : X \to \{-1, +1\}$
- Training error: $\hat{R}(h) = \frac{1}{m} \sum_{i=1}^{m} 1 \left[ h(x_i) \neq y_i \right]$
An alternative derivation of training error

\[ \hat{R}(h) = \frac{1}{m} \sum_{i=1}^{m} 1 \left[ h(x_i) \neq y_i \right] \]
An alternative derivation of training error

\[ \hat{R}(h) = \frac{1}{m} \sum_{i}^{m} \mathbb{1}[h(x_i) \neq y_i] \]  

\[ = \frac{1}{m} \sum_{i}^{m} \begin{cases} 
1 & \text{if } (h(x_i, y_i) == (1, -1) \text{ or } (-1, 1)) \\
0 & \text{otherwise}
\end{cases} \]  

(1)

(2)

(3)

(4)
An alternative derivation of training error

\begin{align*}
\hat{R}(h) &= \frac{1}{m} \sum_{i=1}^{m} \mathbb{1} [h(x_i) \neq y_i] \\
&= \frac{1}{m} \sum_{i=1}^{m} \begin{cases} 
1 & \text{if } (h(x_i, y_i) = (1, -1) \text{ or } (-1, 1)) \\
0 & \text{if } (h(x_i, y_i) = (1, 1) \text{ or } (-1, -1)) 
\end{cases} \\
&= \frac{1}{m} \sum_{i=1}^{m} 1 - y_i h(x_i) \\
&= \frac{1}{m} \sum_{i=1}^{m} \frac{1 - y_i h(x_i)}{2}
\end{align*}
An alternative derivation of training error

\[ \hat{R}(h) = \frac{1}{m} \sum_{i}^{m} \mathbb{1}[h(x_i) \neq y_i] \] (1)

\[ = \frac{1}{m} \sum_{i}^{m} \begin{cases} 1 & \text{if } (h(x_i, y_i) == (1, -1) \text{ or } (-1, 1)) \\ 0 & \text{if } (h(x_i, y_i) == (1, 1) \text{ or } (-1, -1)) \end{cases} \] (2)

\[ = \frac{1}{m} \sum_{i}^{m} \frac{1 - y_i h(x_i)}{2} \] (3)

\[ = \frac{1}{2} - \frac{1}{2m} \sum_{i}^{m} y_i h(x_i) \] (4)
An alternative derivation of training error

\[ \hat{R}(h) = \frac{1}{m} \sum_{i=1}^{m} 1 \left[ h(x_i) \neq y_i \right] \]

\[ = \frac{1}{m} \sum_{i=1}^{m} \begin{cases} 1 & \text{if } (h(x_i, y_i) = (1, -1) \text{ or } (-1, 1)) \\ 0 & \text{if } (h(x_i, y_i) = (1, 1) \text{ or } (-1, -1)) \end{cases} \]

\[ = \frac{1}{m} \sum_{i=1}^{m} \frac{1 - y_i h(x_i)}{2} \]

\[ = \frac{1}{2} - \frac{1}{2m} \sum_{i=1}^{m} y_i h(x_i) \]

Correlation between predictions and labels
An alternative derivation of training error

\[ \hat{R}(h) = \frac{1}{m} \sum_{i} 1 \left[ h(x_i) \neq y_i \right] \]  

(1)

\[ = \frac{1}{m} \sum_{i} \begin{cases} 1 & \text{if } (h(x_i, y_i) = (1, -1) \text{ or } (-1, 1)) \\ 0 & \text{if } (h(x_i, y_i) = (1, 1) \text{ or } (-1, -1)) \end{cases} \]

(2)

\[ = \frac{1}{m} \sum_{i} \frac{1 - y_i h(x_i)}{2} \]

(3)

\[ = \frac{1}{2} - \frac{1}{2m} \sum_{i} y_i h(x_i) \]

(4)

Minimizing training error is thus equivalent to maximizing correlation

\[ \arg \max_{h} \frac{1}{m} \sum_{i} y_i h(x_i) \]

(5)
Imagine where we replace true labels with *Rademacher random variables*

\[ \sigma_i = \begin{cases} +1 \text{ with prob .5} \\ -1 \text{ with prob .5} \end{cases} \]  \hfill (6)
Playing with Correlation

Imagine where we replace true labels with *Rademacher random variables* \( \sigma_i \) where:

\[
\sigma_i = \begin{cases} 
+1 & \text{with prob .5} \\
-1 & \text{with prob .5}
\end{cases}
\]  

This gives us Rademacher correlation—what’s the best that a random classifier could do?

\[
\hat{R}_S(H) = \mathbb{E}_\sigma \left[ \max_{h \in H} \frac{1}{m} \sum_{i=1}^{m} \sigma_i h(x_i) \right]
\]
Imagine where we replace true labels with *Rademacher random variables*

\[ \sigma_i = \begin{cases} +1 & \text{with prob .5} \\ -1 & \text{with prob .5} \end{cases} \]  

This gives us Rademacher correlation—what’s the best that a random classifier could do?

\[
\hat{R}_S(H) \equiv \mathbb{E}_\sigma \left[ \max_{h \in H} \frac{1}{m} \sum_{i=1}^{m} \sigma_i h(x_i) \right]
\]  

Notation: \[ \mathbb{E}_p[f] \equiv \sum_{x} p(x)f(x) \]
Imagine where we replace true labels with *Rademacher random variables* 

\[ \sigma_i = \begin{cases} +1 & \text{with prob } .5 \\ -1 & \text{with prob } .5 \end{cases} \]  

(6)

This gives us Rademacher correlation—what’s the best that a random classifier could do?

\[ \hat{R}_S(H) \equiv \mathbb{E}_\sigma \left[ \max_{h \in H} \frac{1}{m} \sum_{i} \sigma_i h(x_i) \right] \]  

(7)

Note: Empirical Rademacher complexity is with respect to a sample.
Rademacher Extrema

- What are the maximum values of Rademacher correlation?
Rademacher Extrema

- What are the maximum values of Rademacher correlation?

\[ |H| = 1 \quad \text{and} \quad |H| = 2^m \]
Rademacher Extrema

- What are the maximum values of Rademacher correlation?

\[ |H| = 1 \]
\[ \mathbb{E}_{\sigma} \left[ \max_{h \in H} \frac{1}{m} \sum_{i}^{m} \sigma_i h(x_i) \right] \]

\[ |H| = 2^m \]
Rademacher Extrema

- What are the maximum values of Rademacher correlation?

\[
|H| = 1 \\
\mathbb{E}_\sigma \left[ \frac{1}{m} \sum_{i}^{m} \sigma_i \right]
\]

\[
|H| = 2^m
\]

- Rademacher correlation is larger for more complicated hypothesis space.
- What if you’re right for stupid reasons?
Rademacher Extrema

- What are the maximum values of Rademacher correlation?

\[ |H| = 1 \]

\[ h(x_i) \mathbb{E}_{\sigma} \left[ \frac{1}{m} \sum_{i=1}^{m} \sigma_i \right] = 0 \]

- Rademacher correlation is larger for more complicated hypothesis space.

- What if you’re right for stupid reasons?
Rademacher Extrema

- What are the maximum values of Rademacher correlation?

\[ |H| = 1 \]
\[ h(x_i) \mathbb{E}_\sigma \left[ \frac{1}{m} \sum_{i}^{m} \sigma_i \right] = 0 \]

\[ |H| = 2^m \]
\[ \mathbb{E}_\sigma \left[ \max_{h \in H} \frac{1}{m} \sum_{i}^{m} \sigma_i h(x_i) \right] \]
Rademacher Extrema

- What are the maximum values of Rademacher correlation?

| $|H| = 1$ | $|H| = 2^m$ |
|----------------|----------------|
| $h(x_i)\mathbb{E}_{\sigma} \left[ \frac{1}{m} \sum_{i}^{m} \sigma_i \right] = 0$ | $\frac{m}{m} = 1$ |

Rademacher correlation is larger for more complicated hypothesis space.
Rademacher Extrema

- What are the maximum values of Rademacher correlation?

\[ |H| = 1 \]

\[ h(x_i) \mathbb{E}_{\sigma} \left[ \frac{1}{m} \sum_{i}^{m} \sigma_i \right] = 0 \]

\[ |H| = 2^m \]

\[ \frac{m}{m} = 1 \]

- Rademacher correlation is larger for more complicated hypothesis space.

- What if you’re right for stupid reasons?
Generalizing Rademacher Complexity

We can generalize Rademacher complexity to consider all sets of a particular size.

\[ \mathcal{R}_m(H) = \mathbb{E}_{S \sim D^m}[\hat{\mathcal{R}}_S(H)] \] (8)
Theorem

Convergence Bounds Let $F$ be a family of functions mapping from $Z$ to $[0, 1]$, and let sample $S = (z_1, \ldots, z_m)$ were $z_i \sim D$ for some distribution $D$ over $Z$. Define $\mathbb{E}[f] \equiv \mathbb{E}_{z \sim D}[f(z)]$ and $\hat{\mathbb{E}}_S[f] \equiv \frac{1}{m} \sum_{i=1}^{m} f(z_i)$. With probability greater than $1 - \delta$ for all $f \in F$:

$$\mathbb{E}[f] \leq \hat{\mathbb{E}}_S[f] + 2\mathcal{R}_m(F) + O\left(\sqrt{\frac{\ln \frac{1}{\delta}}{m}}\right)$$  (8)
Generalizing Rademacher Complexity

**Theorem**

**Convergence Bounds** Let $F$ be a family of functions mapping from $Z$ to $[0,1]$, and let sample $S = (z_1, \ldots, z_m)$ were $z_i \sim D$ for some distribution $D$ over $Z$. Define $E[f] \equiv E_{z \sim D}[f(z)]$ and $\hat{E}_S[f] \equiv \frac{1}{m} \sum_{i=1}^{m} f(z_i)$. With probability greater than $1 - \delta$ for all $f \in F$:

$$E[f] \leq \hat{E}_S[f] + 2R_m(F) + \mathcal{O}\left(\sqrt{\frac{\ln \frac{1}{\delta}}{m}}\right)$$

(8)

$f$ is a surrogate for the accuracy of a hypothesis (mathematically convenient)
Aside: McDiarmid’s Inequality

If we have a function:

$$|f(x_1, \ldots, x_i, \ldots, x_m) - f(x_1, \ldots, x'_i, \ldots, x_m)| \leq c_i$$

(9)

then:

$$\Pr[f(x_1, \ldots, x_m) \geq \mathbb{E}[f(X_1, \ldots, X_m)] + \epsilon] \leq \exp\left\{ \frac{-2\epsilon^2}{\sum_i c_i^2} \right\}$$

(10)
Aside: McDiarmid’s Inequality

If we have a function:

$$|f(x_1, \ldots, x_i, \ldots x_m) - f(x_1, \ldots, x'_i, \ldots, x_m)| \leq c_i$$ (9)

then:

$$\Pr[f(x_1, \ldots, x_m) \geq \mathbb{E}[f(X_1, \ldots, X_m)] + \epsilon] \leq \exp \left\{ \frac{-2\epsilon^2}{\sum_{i}^m c_i^2} \right\}$$ (10)

Proofs online and in Mohri (requires Martingale, constructing $V_k = \mathbb{E}[V|x_1 \ldots x_k] - \mathbb{E}[V|x_1 \ldots x_{k-1}]$).
Aside: McDiarmid’s Inequality

If we have a function:

$$|f(x_1, \ldots, x_i, \ldots x_m) - f(x_1, \ldots, x'_i, \ldots, x_m)| \leq c_i$$  \hspace{1cm} (9)

then:

$$\Pr[f(x_1, \ldots, x_m) \geq \mathbb{E}[f(X_1, \ldots, X_m)] + \varepsilon] \leq \exp \left\{ \frac{-2\varepsilon^2}{\sum_{i=1}^{m} c_i^2} \right\}$$  \hspace{1cm} (10)

Proofs online and in Mohri (requires Martingale, constructing $V_k = \mathbb{E}[V|x_1 \ldots x_k] - \mathbb{E}[V|x_1 \ldots x_{k-1}]$).

What function do we care about for Rademacher complexity? Let’s define

$$\Phi(S) = \sup_f (\mathbb{E}[f] - \hat{\mathbb{E}}_S[f]) = \sup_f \left( \mathbb{E}[f] - \frac{1}{m} \sum_{i} f(z_i) \right)$$  \hspace{1cm} (11)
Step 1: Bounding divergence from true Expectation

Lemma

**Moving to Expectation** With probability at least $1 - \delta$,

$$\Phi(S) \leq \mathbb{E}_S [\Phi(S)] + \sqrt{\frac{\ln \frac{1}{\delta}}{2m}}$$

Since $f(z_1) \in [0, 1]$, changing any $z_i$ to $z'_i$ in the training set will change $\frac{1}{m} \sum_i f(z_i)$ by at most $\frac{1}{m}$, so we can apply McDiarmid’s inequality with

$$\epsilon = \sqrt{\frac{\ln \frac{1}{\delta}}{2m}} \text{ and } c_i = \frac{1}{m}.$$
Step 2: Comparing two different empirical expectations

Define a ghost sample \( S' = (z'_1, \ldots, z'_m) \sim D \). How much can two samples from the same distribution vary?

**Lemma**

**Two Different Samples**

\[
\mathbb{E}_S[\Phi(S)] = \mathbb{E}_S \left[ \sup_f (\mathbb{E}[f] - \hat{\mathbb{E}}_S[f]) \right] \quad (12)
\]

\[
(13)
\]
Step 2: Comparing two different empirical expectations

Define a ghost sample $S' = (z'_1, \ldots, z'_m) \sim D$. How much can two samples from the same distribution vary?

**Lemma**

**Two Different Samples**

\[
E_S[\Phi(S)] = E_S \left[ \sup_f (E[f] - \hat{E}_S[f]) \right]
\]

\[
= E_S \left[ \sup_{f \in F} (E_{S'}[\hat{E}_{S'}[f]] - \hat{E}_S[f]) \right]
\]

The expectation is equal to the expectation of the empirical expectation of all sets $S'$
Step 2: Comparing two different empirical expectations

Define a ghost sample $S' = (z'_1, \ldots, z'_m) \sim D$. How much can two samples from the same distribution vary?

**Lemma**

**Two Different Samples**

$$
\mathbb{E}_S[\Phi(S)] = \mathbb{E}_S \left[ \sup_{f} (\mathbb{E}[f] - \hat{\mathbb{E}}_S[f]) \right] 
$$

$$
= \mathbb{E}_S \left[ \sup_{f \in F} (\mathbb{E}_{S'}[\hat{\mathbb{E}}_{S'}[f]] - \hat{\mathbb{E}}_S[f]) \right] 
$$

$$
= \mathbb{E}_S \left[ \sup_{f \in F} (\mathbb{E}_{S'}[\hat{\mathbb{E}}_{S'}[f]] - \hat{\mathbb{E}}_S[f]) \right] 
$$

$S$ and $S'$ are distinct random variables, so we can move inside the expectation.
Step 2: Comparing two different empirical expectations

Define a ghost sample $S' = (z'_1, \ldots, z'_m) \sim D$. How much can two samples from the same distribution vary?

**Lemma**

**Two Different Samples**

$$
\mathbb{E}_S[\Phi(S)] = \mathbb{E}_S\left[\sup_f (\mathbb{E}[f] - \hat{\mathbb{E}}_S[f])\right] \quad (12)
$$

$$
= \mathbb{E}_S\left[\sup_{f \in F} (\mathbb{E}_{S'}[\hat{\mathbb{E}}_{S'}[f] - \hat{\mathbb{E}}_S[f]])\right] \quad (13)
$$

$$
\leq \mathbb{E}_{S,S'}\left[\sup_f (\hat{\mathbb{E}}_{S'}[f] - \hat{\mathbb{E}}_S[f])\right] \quad (14)
$$

The expectation of a max over some function is at least the max of that expectation over that function.
Step 3: Adding in Rademacher Variables

From $S, S'$ we’ll create $T, T'$ by swapping elements between $S$ and $S'$ with probability $.5$. This is still independent, identically distributed (iid) from $D$. They have the same distribution:

$$\hat{E}_{S'}[f] - \hat{E}_S[f] \sim \hat{E}_{T'}[f] - \hat{E}_T[f]$$  \hspace{1cm} (15)
Step 3: Adding in Rademacher Variables

From $S, S'$ we’ll create $T, T'$ by swapping elements between $S$ and $S'$ with probability .5. This is still independent, identically distributed (iid) from $D$. They have the same distribution:

$$\hat{E}_{S'}[f] - \hat{E}_S[f] \sim \hat{E}_{T'}[f] - \hat{E}_T[f]$$  \hspace{1cm} (15)

Let’s introduce $\sigma_i$:

$$\hat{E}_{T'}[f] - \hat{E}_T[f] = \frac{1}{m} \left\{ \begin{array}{ll} f(z_i) - f(z_i') & \text{with prob .5} \\ f(z_i') - f(z_i) & \text{with prob .5} \end{array} \right.$$  \hspace{1cm} (16)

$$= \frac{1}{m} \sum_i \sigma_i (f(z_i') - f(z_i))$$  \hspace{1cm} (17)
Step 3: Adding in Rademacher Variables

From $S, S'$ we’ll create $T, T'$ by swapping elements between $S$ and $S'$ with probability $0.5$. This is still independent, identically distributed (iid) from $D$. They have the same distribution:

$$
\hat{E}_{S'}[f] - \hat{E}_S[f] \sim \hat{E}_{T'}[f] - \hat{E}_T[f] \tag{15}
$$

Let’s introduce $\sigma_i$:

$$
\hat{E}_{T'}[f] - \hat{E}_T[f] = \frac{1}{m} \begin{cases} 
    f(z_i) - f(z'_i) & \text{with prob } 0.5 \\
    f(z'_i) - f(z_i) & \text{with prob } 0.5
\end{cases} \tag{16}
$$

$$
= \frac{1}{m} \sum_i \sigma_i (f(z'_i) - f(z_i)) \tag{17}
$$

Thus:

$$
\mathbb{E}_{S,S'} \left[ \sup_{f \in F} (\hat{E}_{S'}[f] - \hat{E}_S[f]) \right] = \mathbb{E}_{S,S',\sigma} \left[ \sup_{f \in F} \left( \sum_i \sigma_i (f(z'_i) - f(z_i)) \right) \right].
$$
Step 4: Making These Rademacher Complexities

Before, we had $\mathbb{E}_{S, S', \sigma} \left[ \sup_{f \in F} \sum \sigma_i (f(z') - f(z)) \right]$
Step 4: Making These Rademacher Complexities

Before, we had:

$$\mathbb{E}_{S,S',\sigma} \left[ \sup_{f \in F} \sum_i \sigma_i (f(z'_i) - f(z_i)) \right]$$

$$\leq \mathbb{E}_{S,S',\sigma} \left[ \sup_{f \in F} \sum_i \sigma_i f(z'_i) + \sup_{f \in F} \sum_i (-\sigma_i) f(z_i) \right]$$  \hspace{1cm} (18)

Taking the sup jointly must be less than or equal the individual sup.
Step 4: Making These Rademacher Complexities

Before, we had $\mathbb{E}_{S,S',\sigma} \left[ \sup_{f \in F} \sum_i \sigma_i (f(z'_i) - f(z_i)) \right]$

\begin{align*}
\leq & \mathbb{E}_{S,S',\sigma} \left[ \sup_{f \in F} \sum_i \sigma_i f(z'_i) + \sup_{f \in F} \sum_i (-\sigma_i) f(z_i) \right] \tag{18} \\
\leq & \mathbb{E}_{S,S',\sigma} \left[ \sup_{f \in F} \sum_i \sigma_i f(z'_i) \right] + \mathbb{E}_{S,S',\sigma} \left[ \sup_{f \in F} \sum_i (-\sigma_i) f(z_i) \right] \tag{19} \\
\end{align*}

Linearity
Step 4: Making These Rademacher Complexities

Before, we had $\mathbb{E}_{S,S',\sigma} \left[ \sup_{f \in F} \sum_i \sigma_i (f(z'_i) - f(z_i)) \right]$.

\begin{align*}
\leq & \mathbb{E}_{S,S',\sigma} \left[ \sup_{f \in F} \sum_i \sigma_i f(z'_i) + \sup_{f \in F} \sum_i (-\sigma_i) f(z_i) \right] \quad (18) \\
\leq & \mathbb{E}_{S,S',\sigma} \left[ \sup_{f \in F} \sum_i \sigma_i f(z'_i) \right] + \mathbb{E}_{S,S',\sigma} \left[ \sup_{f \in F} \sum_i (-\sigma_i) f(z_i) \right] \quad (19) \\
= & \mathcal{R}_m(F) + \mathcal{R}_m(F) \quad (20)
\end{align*}

Definition
Putting the Pieces Together

With probability \( \geq 1 - \delta \):

\[
\Phi(S) \leq \mathbb{E}_S[\Phi(S)] + \sqrt{\frac{\ln \frac{1}{\delta}}{2m}}
\]  

(21)

Step 1
Putting the Pieces Together

With probability $\geq 1 - \delta$:

$$\sup_{f} (\mathbb{E}[f] - \hat{\mathbb{E}}_{S}[h]) \leq \mathbb{E}_{S}[\Phi(S)] + \sqrt{\frac{\ln \frac{1}{\delta}}{2m}}$$

(21)

Definition of $\Phi$
Putting the Pieces Together

With probability $\geq 1 - \delta$:

$$
E[f] - \hat{E}_S[h] \leq E_S[\Phi(S)] + \sqrt{\frac{\ln \frac{1}{\delta}}{2m}}
$$

Drop the sup, still true
Putting the Pieces Together

With probability \( \geq 1 - \delta \):

\[
\mathbb{E}[f] - \hat{E}_S[h] \leq \mathbb{E}_{S,S'} \left[ \sup_f (\hat{E}_{S'}[f] - \hat{E}_S[f]) \right] + \sqrt{\frac{\ln \frac{1}{\delta}}{2m}} \tag{21}
\]

Step 2
Putting the Pieces Together

With probability $\geq 1 - \delta$:

$$
\mathbb{E}[f] - \hat{\mathbb{E}}_S[h] \leq \mathbb{E}_{S,S',\sigma} \left[ \sup_{f \in F} \left( \sum_i \sigma_i (f(z'_i) - f(z_i)) \right) \right] + \sqrt{\frac{\ln \frac{1}{\delta}}{2m}} \quad (21)
$$

Step 3
Putting the Pieces Together

With probability $\geq 1 - \delta$:

$$\mathbb{E}[f] - \hat{\mathbb{E}}_S[h] \leq 2\mathcal{R}_m(F) + \sqrt{\frac{\ln \frac{1}{\delta}}{2m}}$$

(21)

Step 4
Putting the Pieces Together

With probability $\geq 1 - \delta$:

$$
\mathbb{E}[f] - \hat{\mathbb{E}}_S[h] \leq 2\mathcal{R}_m(F) + \sqrt{\frac{\ln \frac{1}{\delta}}{2m}}
$$

(21)

Recall that $\hat{\mathcal{R}}_S(F) \equiv \mathbb{E}_\sigma \left[ \sup_f \frac{1}{m} \sum_i \sigma_i f(z_i) \right]$, so we apply McDiarmid’s inequality again (because $f \in [0,1]$):

$$
\hat{\mathcal{R}}_S(F) \leq \mathcal{R}_m(F) + \sqrt{\frac{\ln \frac{1}{\delta}}{2m}}
$$

(22)
Putting the Pieces Together

With probability $\geq 1 - \delta$:

$$\mathbb{E} [f] - \hat{\mathbb{E}}_S [h] \leq 2\mathcal{R}_m (F) + \sqrt{\frac{\ln \frac{1}{\delta}}{2m}}$$

(21)

Recall that $\hat{\mathcal{R}}_S (F) \equiv \mathbb{E}_\sigma \left[ \sup_f \frac{1}{m} \sum_i \sigma_i f(z_i) \right]$, so we apply McDiarmid’s inequality again (because $f \in [0, 1]$):

$$\hat{\mathcal{R}}_S (F) \leq \mathcal{R}_m (F) + \sqrt{\frac{\ln \frac{1}{\delta}}{2m}}$$

(22)

Putting the two together:

$$\mathbb{E} [f] \leq \hat{\mathbb{E}}_S [f] + 2\mathcal{R}_m (F) + O \left( \sqrt{\frac{\ln \frac{1}{\delta}}{m}} \right)$$

(23)
What about hypothesis classes?

Define:

\[ Z \equiv X \times \{-1, +1\} \quad (24) \]

\[ f_h(x, y) \equiv 1 \left[ h(x) \neq y \right] \quad (25) \]

\[ F_H \equiv \{f_h : h \in H\} \quad (26) \]
What about hypothesis classes?

Define:

\[ Z \equiv X \times \{-1, +1\} \]  \hspace{2cm} (24)

\[ f_h(x, y) \equiv 1 \left[ h(x) \neq y \right] \]  \hspace{2cm} (25)

\[ F_H \equiv \{ f_h : h \in H \} \]  \hspace{2cm} (26)

We can use this to create expressions for generalization and empirical error:

\[ R(h) = \mathbb{E}_{(x,y) \sim D} \left[ 1 \left[ h(x) \neq y \right] \right] = \mathbb{E} \left[ f_h \right] \]  \hspace{2cm} (27)

\[ \hat{R}(h) = \frac{1}{m} \sum_i 1 \left[ h(x_i) \neq y \right] = \hat{\mathbb{E}}_S \left[ f_h \right] \]  \hspace{2cm} (28)
What about hypothesis classes?

Define:

\[ Z \equiv X \times \{-1, +1\} \]  
\[ f_h(x, y) \equiv 1 \left[ h(x) \neq y \right] \]  
\[ F_H \equiv \{ f_h : h \in H \} \]

We can use this to create expressions for generalization and empirical error:

\[ R(h) = \mathbb{E}_{(x, y) \sim D} \left[ 1 \left[ h(x) \neq y \right] \right] = \mathbb{E} [f_h] \]  
\[ \hat{R}(h) = \frac{1}{m} \sum_i 1 \left[ h(x_i) \neq y \right] = \hat{\mathbb{E}}_S [f_h] \]

We can plug this into our theorem!
Generalization bounds

- We started with expectations

\[
\mathbb{E} [f] \leq \hat{\mathbb{E}}_S [f] + 2 \hat{\mathcal{R}}_S (F) + O \left( \sqrt{\frac{\ln \frac{1}{\delta}}{m}} \right)
\]  

(29)

- We also had our definition of the generalization and empirical error:

\[
R(h) = \mathbb{E}_{(x,y) \sim \mathcal{D}} [\mathbb{1} [h(x) \neq y]] = \mathbb{E} [f_h]
\]

\[
\hat{R}(h) = \frac{1}{m} \sum_i \mathbb{1} [h(x_i) \neq y] = \hat{\mathbb{E}}_S [f_h]
\]

- Combined with the previous result:

\[
\hat{\mathcal{R}}_S (F_H) = \frac{1}{2} \hat{\mathcal{R}}_S (H)
\]  

(30)

- All together:

\[
R(h) \leq \hat{R}(h) + \mathcal{R}_m (H) + O \left( \sqrt{\frac{\log \frac{1}{\delta}}{m}} \right)
\]  

(31)
Wrapup

- Interaction of data, complexity, and accuracy
- Still very theoretical
- Next up: How to evaluate generalizability of specific hypothesis classes
Recap

• Rademacher complexity provides nice guarantees

\[ R(h) \leq \hat{R}(h) + \mathcal{R}_m(H) + O(\sqrt{\frac{\log \frac{1}{\delta}}{2m}}) \] (32)

• But in practice hard to compute for real hypothesis classes
• Is there a relationship with simpler combinatorial measures?
Growth Function

Define the growth function $\Pi_H : \mathbb{N} \rightarrow \mathbb{N}$ for a hypothesis set $H$ as:

$$\forall m \in \mathbb{N}, \Pi_H(m) \equiv \max_{\{x_1, \ldots, x_m\} \in X} \left| \{(h(x_1), \ldots, h(x_m) : h \in H)\} \right|$$

(33)
Growth Function

Define the **growth function** \( \Pi_H : \mathbb{N} \rightarrow \mathbb{N} \) for a hypothesis set \( H \) as:

\[
\forall m \in \mathbb{N}, \, \Pi_H(m) \equiv \max_{\{x_1, \ldots, x_m\} \in X} \left| \{(h(x_1), \ldots, h(x_m) : h \in H}\right| \tag{33}
\]

i.e., the number of ways \( m \) points can be classified using \( H \).
Rademacher Complexity vs. Growth Function

If $G$ is a function taking values in $\{-1, +1\}$, then

$$R_m(G) \leq \sqrt{\frac{2 \ln \Pi_G(m)}{m}}$$  \hspace{1cm} (34)

Uses Masar’s lemma
Vapnik-Chervonenkis Dimension

\[ \text{VC}(H) \equiv \max \{ m : \Pi_H(m) = 2^m \} \]  (35)
Vapnik-Chervonenkis Dimension

The size of the largest set that can be fully shattered by $H$.

$$VC(H) \equiv \max \left\{ m : \Pi_H(m) = 2^m \right\}$$  (35)
VC Dimension for Hypotheses

- Need upper and lower bounds
- Lower bound: example
- Upper bound: Prove that no set of \(d + 1\) points can be shattered by \(H\) (harder)
What is the VC dimension of \([a, b]\) intervals on the real line.
Intervals

What is the VC dimension of $[a, b]$ intervals on the real line.

- What about two points?

- Two points can be perfectly classified, so VC dimension $\geq 2$

- Thus, VC dimension of intervals is 2
Intervals

What is the VC dimension of \([a, b]\) intervals on the real line.

- What about two points?

![Diagram showing intervals and classification examples]
Intervals

What is the VC dimension of \([a, b]\) intervals on the real line.

- What about two points?

\[ \begin{align*}
\text{\includegraphics[width=\textwidth]{interval_diagram.png}}
\end{align*} \]
Intervals

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- What about three points?
Intervals

What is the VC dimension of \([a, b]\) intervals on the real line.

- Two points can be perfectly classified, so VC dimension \(\geq 2\)
- What about three points?

![Diagram of intervals on the real line with two green plus signs and one red interval]

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Intervals

What is the VC dimension of \([a, b]\) intervals on the real line.

- Two points can be perfectly classified, so VC dimension \(\geq 2\)
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- **No set** of three points can be shattered
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Sine Functions

- Consider hypothesis that classifies points on a line as either being above or below a sine wave
  \[ \{ t \mapsto \sin(\omega x) : \omega \in \mathbb{R} \} \] (36)
- Can you shatter three points?

• Can you shatter four points?
• How many points can you shatter?

Thus, VC dim of sine on line is \( \infty \)
Sine Functions

- Consider hypothesis that classifies points on a line as either being above or below a sine wave

$$\{ t \rightarrow \sin(\omega x) : \omega \in \mathbb{R} \}$$ (36)

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Connecting VC with growth function

VC dimension obviously encodes the complexity of a hypothesis class, but we want to connect that to Rademacher complexity and the growth function so we can prove generalization bounds.
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**Theorem**

**Sauer’s Lemma** Let $H$ be a hypothesis set with VC dimension $d$. Then

$$\forall m \in \mathbb{N}$$

$$\Pi_H(m) \leq \sum_{i=0}^{d} \binom{m}{i} \equiv \Phi_d(m)$$  \hspace{1cm} (37)
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$$\forall m \in \mathbb{N}$$

$$\Pi_H(m) \leq \sum_{i=0}^{d} \binom{m}{i} \equiv \Phi_d(m) \quad (37)$$

This is good because the sum when multiplied out becomes

$$\binom{m}{i} = \frac{m(m-1)\ldots}{i!} = \Theta(m^d)$$

When we plug this into the learning error limits:

$$\log(\Pi_H(2m)) = \log(\Theta(m^d)) = \Theta(d \log m).$$
Proof of Sauer’s Lemma

Prelim:

\[
\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1} \quad \text{This comes from Pascal’s Triangle}
\]

\[
\binom{m}{k} = 0 \quad \text{if} \quad \begin{cases} 
  k < 0 \\
  k > m
\end{cases} \quad \text{This convention is consistent with Pascal’s Triangle}
\]
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\[
\binom{m}{k} = 0 \quad \text{if} \quad \begin{cases} k < 0 \\ k > m \end{cases} \quad \text{This convention is consistent with Pascal’s Triangle}
\]

We’ll proceed by induction. Our two base cases are:

- If \( m = 0 \), \( \Pi_H(m) = 1 \). You have no data, so there’s only one (degenerate) labeling
- If \( d = 0 \), \( \Pi_H(m) = 1 \). If you can’t even shatter a single point, then it’s a fixed function
Induction Step

Assume that it holds for all $m', d'$ for which $m' + d' < m + d$. We are given $H, |S| = m$, $S = \langle x_1, \ldots, x_m \rangle$, and $d$ is the VC dimension of $H$. 
Induction Step

Assume that it holds for all $m', d'$ for which $m' + d' < m + d$. We are given $H, |S| = m$, $S = \langle x_1, \ldots, x_m \rangle$, and $d$ is the VC dimension of $H$.

**Build two new hypothesis spaces**

<table>
<thead>
<tr>
<th>$\mathcal{H}$</th>
<th>$\mathcal{H}_1$</th>
<th>$\mathcal{H}_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1, \ldots, x_m$</td>
<td>$x_1, \ldots, x_{m-1}$</td>
<td>$x_1, \ldots, x_{m-1}$</td>
</tr>
<tr>
<td>h1 0 1 1 0 0</td>
<td>h1 0 1 1 0</td>
<td>h1 0 1 1 0</td>
</tr>
<tr>
<td>h2 0 1 1 0 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>h3 0 1 1 1 0</td>
<td>h3 0 1 1 1</td>
<td></td>
</tr>
<tr>
<td>h4 1 0 0 1 0</td>
<td>h4 1 0 0 1</td>
<td>h4 1 0 0 1</td>
</tr>
<tr>
<td>h5 1 0 0 1 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>h6 1 1 0 0 1</td>
<td>h6 1 1 0 0</td>
<td></td>
</tr>
</tbody>
</table>

Encodes where the extended set has differences on the first $m$ points.
Bounding Growth Function

\[ |\Pi_H(S)| = |H_1| + |H_2| \]

\[ \leq \sum_{i=0}^{d} \binom{m-1}{i} + \sum_{i=0}^{d-1} \binom{m-1}{i} \]

(38)  
(39)  
(40)
Bounding Growth Function

\[ |\Pi_H(S)| = |H_1| + |H_2| \leq \sum_{i=0}^{d} \binom{m-1}{i} + \sum_{i=0}^{d-1} \binom{m-1}{i} \]

We can rewrite this as \( \sum_{i=0}^{d} \binom{m-1}{i-1} \) because \( \binom{x}{-1} = 0 \).
Bounding Growth Function

\[ |\Pi_H(S)| = |H_1| + |H_2| \]  \hspace{1cm} (38)

\[ \sum_{i=0}^{d} \binom{m-1}{i} + \sum_{i=0}^{d-1} \binom{m-1}{i} \]  \hspace{1cm} (39)

\[ = \sum_{i=0}^{d} \left[ \binom{m-1}{i} + \binom{m-1}{i-1} \right] \]  \hspace{1cm} (40)

\[ = \sum_{i=0}^{d-1} \binom{m-1}{i} + \binom{m-1}{d-1} \]  \hspace{1cm} (41)
Bounding Growth Function

$$|\Pi_H(S)| = |H_1| + |H_2|$$  \hspace{1cm} (38)

$$\leq \sum_{i=0}^{d} \binom{m-1}{i} + \sum_{i=0}^{d-1} \binom{m-1}{i}$$  \hspace{1cm} (39)

$$= \sum_{i=0}^{d} \left[ \binom{m-1}{i} + \binom{m-1}{i-1} \right]$$  \hspace{1cm} (40)

$$= \sum_{i=0}^{d} \binom{m}{i}$$  \hspace{1cm} (41)

(42)

Pascal’s Triangle
Bounding Growth Function

\[ |\Pi_H(S)| = |H_1| + |H_2| \]  

\[ \leq \sum_{i=0}^{d} \binom{m-1}{i} + \sum_{i=0}^{d-1} \binom{m-1}{i} \]  

\[ = \sum_{i=0}^{d} \left[ \binom{m-1}{i} + \binom{m-1}{i-1} \right] \]  

\[ = \sum_{i=0}^{d} \binom{m}{i} \]  

\[ = \Phi_d(m) \]
Wait a minute . . .

Is this combinatorial expression really $O(m^d)$?

\[
\sum_{i=0}^{d} \binom{m}{i} \leq \sum_{i=0}^{d} \binom{m}{i} \left(\frac{m}{d}\right)^{d-i} \\
\leq \sum_{i=0}^{m} \binom{m}{i} \left(\frac{m}{d}\right)^{d-i} \\
= \left(\frac{m}{d}\right)^d \sum_{i=0}^{m} \binom{m}{i} \left(\frac{d}{m}\right)^i \\
= \left(\frac{m}{d}\right)^d \left(1 + \frac{d}{m}\right)^m \leq \left(\frac{m}{d}\right)^d e^d.
\]
Combining our previous generalization results with Sauer’s lemma, we have that for a hypothesis class $H$ with VC dimension $d$, for any $\delta > 0$ with probability at least $1 - \delta$, for any $h \in H$,

$$R(h) \leq \hat{R}(h) + \sqrt{\frac{2d \log \frac{em}{d}}{m}} + \sqrt{\frac{\log \frac{1}{\delta}}{2m}}$$

(43)
Whew!

- We now have some theory down
- We’re now going to see if we can find an algorithm that has good VC dimension
Whew!

- We now have some theory down
- We’re now going to see if we can find an algorithm that has good VC dimension
- And works well in practice . . .
Whew!

- We now have some theory down
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- And works well in practice . . . Support Vector Machines
• We now have some theory down
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• And works well in practice . . . Support Vector Machines
• In class: more VC dimension examples