Beyond worst-case analysis: Observed low depth for a P-complete problem

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Design and analysis of algorithms: core theory topic to CS majors

- Nearly all examples in popular textbooks involve:
  - Design for worst case performance, and
  - Analyze for the worst case

<u>Take home message from "theory"</u>

Worst-case performance is the silver bullet

In particular: NP-Complete ➔

Problem (as-is) is

Surprise 1: Practice is… "misbehaving"

- Some of the most commercially impactful algorithms do not fit the worst case mold
- Examples Algorithms for:
  - Clustering
  - Linear programming
  - Neural network training
  - SAT solvers (quintessential NP-complete problem)
- Metric: *wall-clock time*
- Beyond Worst-Case (BWC): NP-complete?... let's see…
- Commended (concerted) attempt: Roughgarden, T., Editor (2021). Beyond the Worst-Case Analysis of Algorithms. Cambridge University Press. 30 chapters.

Upshot for "theory":

Worst-case? …You risk missing big impact opportunities

Surprise 2: No parallel algorithms in edited book

- Since 2005, parallelism leads performance growth opportunities
- Worst-case is mostly about performance, and so is …BWC…
- Still: no parallel algorithms in the edited book
  - Note: parallel matrix multiplication - a heavily used routine. BUT in a worst-case form. Only the (serial) calls to the routine are where BWC tends to happen
- Let's ponder: how come? how could BWC application emerge? Wall clock trial&error by parallel algorithmicists and programmers

Surprise (?) 3: No commercial "general-purpose" manycores.
Too few program or taught to program in parallel.

[Beyond current scope: Vendors' fault or theory fault?]

In any case, where could empirical evidence emerge from?

Back to complexity theory

- NP-complete ➔ “no” polynomial time serial algorithm. Still, practical BWC algorithms for some NP-complete problems
- P-Complete➔”no” parallel algorithm with poly-log worst-case time & poly processor#…

  “nightmare for parallel processing”

  [MehlhornSanders08]

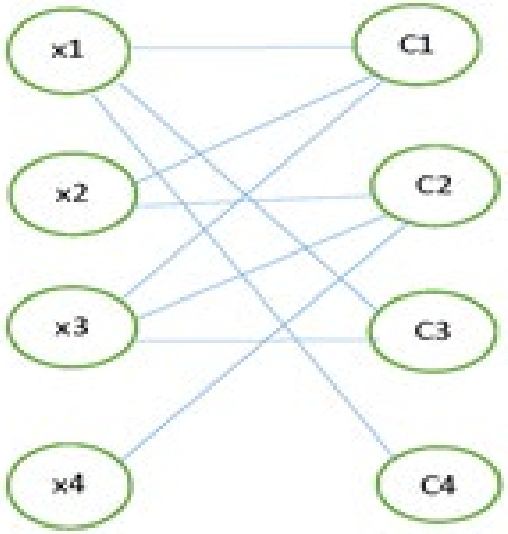
- BWC: ?

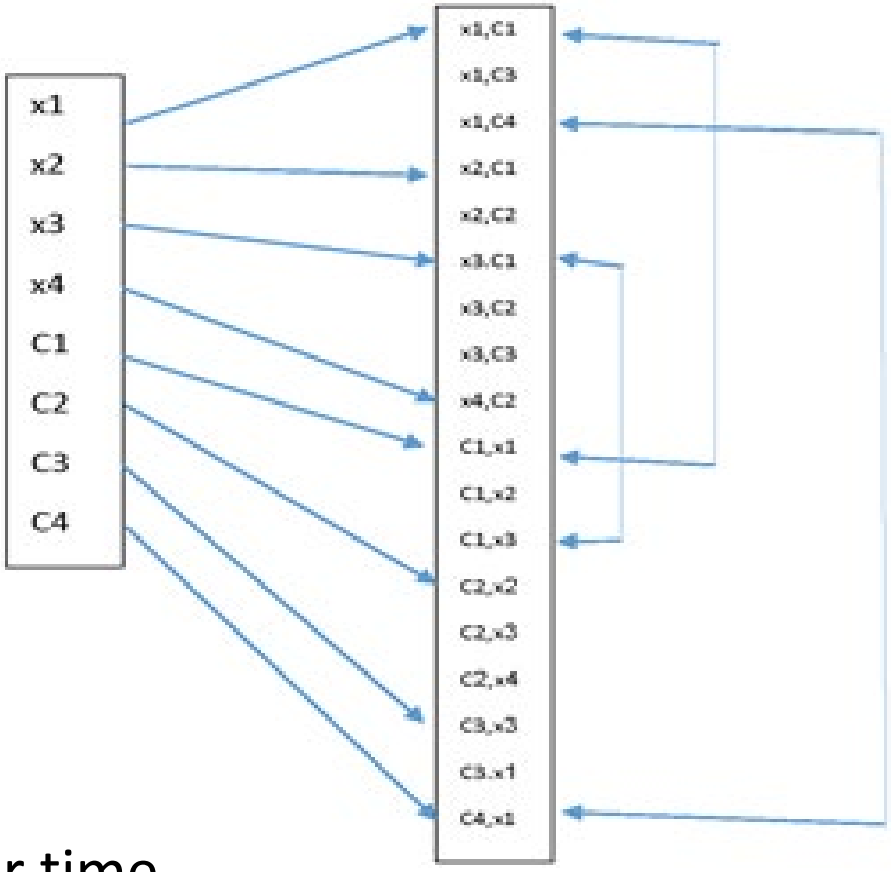
Surprise 4: A first very efficient BWC parallel algorithm for a P-Complete problem—The **current paper**

- Wait, didn’t you just tell us that BWC correlates with wall clock time, but the manycore needed is not available?
- Generally, a chicken-and-egg impasse, but the paper circumvents it by resorting to a model. Hence: PMAM.

3 CNF Horn-SAT

Example: C1 = (x1 V ¬x2 V ¬x3), C2 = (x2 V ¬x3 V ¬x4), C3 = (x3 V ¬x1), C4 = (x1)





Per clause: ≤3 literals. ≤1 positive.

Formula: C1∧C2∧C3∧C4

Problem: is the formula satisfiable?

Known: 3 CNF Horn-SAT: (i) has linear time serial algorithm, but (ii) is P-Complete

Loop of the parallel algorithm: "parallel unit propagation"

While there are clauses with singleton literals ("unit clauses")

Satisfy all of these literals

In parallel, "Remove":

(i) all satisfied clauses, and

(ii) all implied FALSE literals in every clause in which they appear.

For every affected clause, perform the following case analysis:

a. No literals remain. Conclude: the formula is unsatisfiable. Terminate the algorithm.

b. Only one literal remained. Include this literal in the set of singleton literals for the next iteration

<u>Final step</u>. If the above did not conclude that the formula is satisfiable or unsatisfiable, conclude: The formula is satisfiable. Assign FALSE to all remaining variables to derive a satisfying assignment.

N – total number of literals.

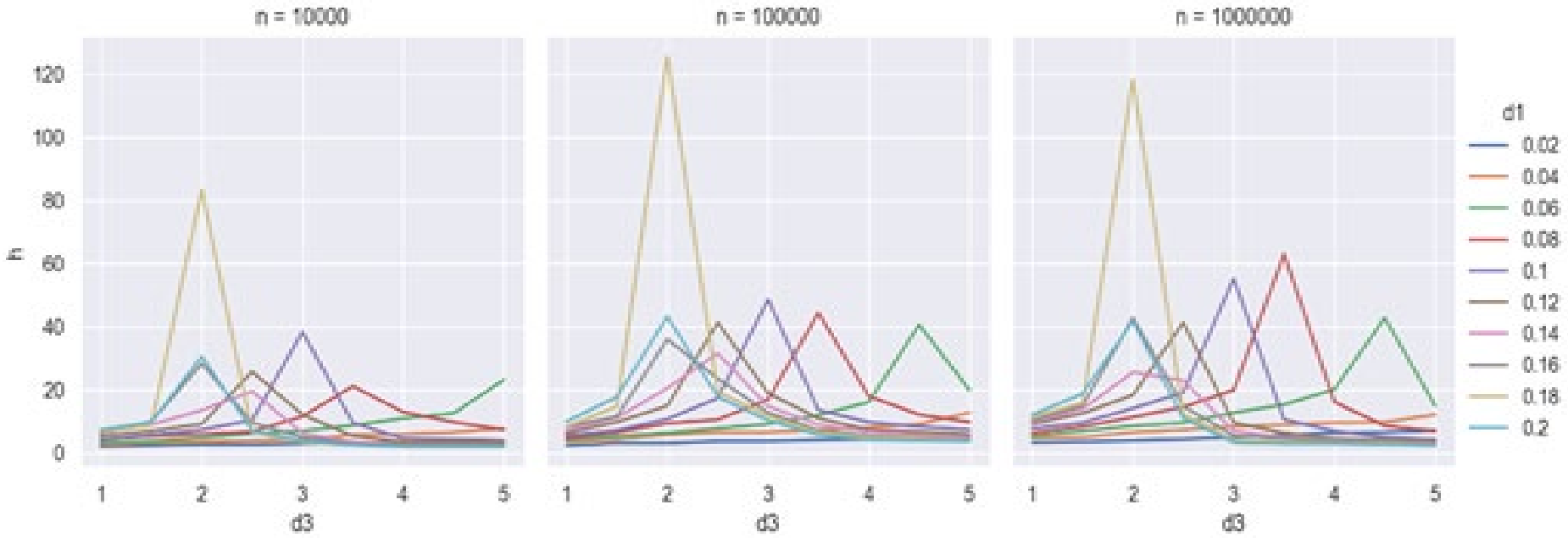
h – number of iterations. Namely: length of the longest chain of such commit iterations, given the input formula (by limited analogy to BFS).

On a PRAM: O(N/p + h log N) time or O(N/p) time for p <= N/ (h log n)

Recall: Circumvent chicken&egg? **h dominates**. So, just observe values of h.

Random Horn SAT formulae model [Moore 2007]

Given d1 < 1 and d3, the random Horn-SAT formula H(N,d1,d3) is the conjunction of:
(1) a single negative literal x1; (2) d1N positive literals chosen uniformly from variables x2, …, xN;
(3) d3N 3-literal clauses. 3 variables are chosen uniformly. One positive literal.
Experiments 3 values for N: 10,000, 100,000, 1,000,000. 02 ≤ d1 ≤ 0.2, 1 ≤ d3 ≤ 5.
Figure shows: average value of h exceeded 60 only for one category: d3=2 and d1=0.18.

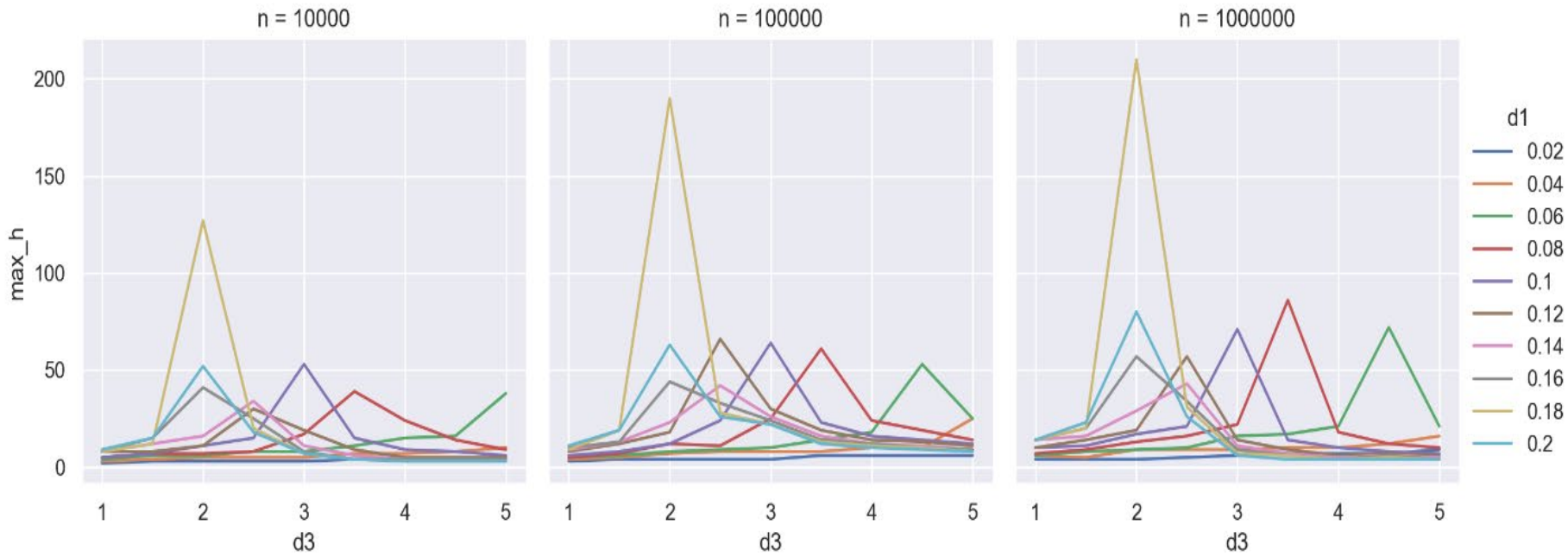


Average value of h for each category (number of variables, d1 and d3)

Largest observed value of h

Figure shows: largest value observed was slightly above 200 for the only category where the average value of h exceeded 60.

Note: for N=1,000,000, h is lower than N by nearly 4 order of magnitude or more ➔ low parallel time. Noteworthy for our P-complete problem.



Largest observed value of h for each category (number of variables, d1 and d3)

Horn SAT formulae derived from an application

- SAT formulae models a question (e.g., correctness, reachability) reduced to satisfiability
- Results are even more appealing for h for such an application
- For instances that led to over 100M variables the largest h observed was only 18. Hardly a nightmare…

Conclusion

- The current paper used an ad-hoc model for: theory nightmare ➔ wake up to rosy BWC outlook. But, essentially it is a PRAM algorithm
- Prior UMD “PRAM-On-Chip” HW-SW prototyping
- IMO, HW vendors bungled up so far the general-purpose manycore opportunity
- Competition among CPU vendors is becoming fierce again for the first time since the 1990s
- How to get to general-purpose manycore is beyond scope for current paper