INCIDENT FIELD RECOVERY FOR AN ARBITRARY-SHAPED SCATTERER

*Dmitry N. Zotkin, Nail A. Gumerov, and Ramani Duraiswami**

VisiSonics Corporation, College Park, MD 20742
{dmitry.z,nail.g,ramani.d}@visisonics.com

ABSTRACT

Any acoustic sensor disturbs the spatial acoustic field to certain extent, and a recorded field is different from a field that would have existed if a sensor were absent. Recovery of the original (incident) field is a fundamental task in spatial audio. For some sensor geometries, the disturbance of the field by the sensor can be characterized analytically and its influence can be undone; however, for arbitrary-shaped sensor numerical methods have to be employed. In the current work, the sensor influence on the field is characterized using numerical (specifically, boundary-element) methods, and a framework to recover the incident field, either in the plane-wave or in the spherical wave function basis, is developed. Field recovery in terms of the spherical basis allows the generation of a higher-order Ambisonics representation of the spatial audio scene. Experimental results using a complex-shaped scatterer are presented.

Index Terms— Ambisonics, acoustic fields, boundary-element methods, head-related transfer function, spatial audio.

1. INTRODUCTION

Spatial audio reproduction is an ability to deliver to the listener an immersive sense of presence in an acoustic scene as if they were actually there. Such delivery can be performed via a distributed set of loudspeakers or via headphones, and the scene presented could be either synthetic (created from scratch using individual audio stems), real (recorded using a spatial audio recording apparatus), or augmented (using real as a base and adding a number of synthetic components). This work is focused on designing such recording apparatus well-suited for capturing the spatial structure of the audio field for above-mentioned immersive reproduction or for other purposes such as auditory scene analysis and understanding.

Any measurement device disturbs, to some degree, the process being measured. A single small microphone offers the least degree of disturbance but is unable to capture the spatial structure of the acoustic field. A large number of microphones randomly distributed over the space of interest are able to sample the field spatial structure very well; however, microphones must be physically supported by rigid hardware, and designing it in a way so as not to disturb the sound field is difficult; furthermore, the differences in sampling locations requires analysis to obtain the sound field at a region of interest. One well-known solution is to shape a microphone support in a way so that the support's influence can be computed analytically (e.g. to use a spherical baffle) and factored out. This solution is feasible; however, in many cases the support geometry is irregular and is constrained by external factors. As an example, one can

think of an anthropomorphic (or a quadruped) robot, whose body shape is dictated by a required functionality and/or appearance and microphones must be mounted on the body surface.

In the current work, a method to factor out the contribution of *arbitrary* support and to recover the field at specified points as it would be if the support were absent is proposed. The method is based on numerically computing the transfer function between the incident plane wave (input) and the signal recorded by a microphone mounted on support (output) as a function of plane wave direction and microphone location (as plane waves form complete basis over the sphere and Helmholtz equation is linear, arbitrary input can be represented in this basis with some weights and output for said input can be obtained by combining outputs for every basis function with the same weights). Such a transfer function is similar to the head-related transfer function (HRTF) and it will be called "HRTF" in this work for the sake of simplicity (although an arbitrary-shaped support is used and no "head" is involved; note that the HRTF itself is somewhat of a misnomer as other parts of the body, notably the pinnae and shoulders, also contribute to sound scattering). Further, having the HRTF available and given the pressure measured at microphones, the set of plane-wave coefficients that best describes the incident field is found using a least-squares solution.

Another complete basis over the sphere is the set of spherical wave functions (SH). Just like the HRTF is a potential generated by a single basis function (plane wave) at the location of the microphone, an HRTF-like function can be introduced that describes the potential created at the microphone location by an incident field comprised of a single spherical wave function. This approach offers computational advantages for deriving HRTF numerically; also, it naturally leads to a framework for computing incident field representation in terms of the SH basis, which is used in the current work to record incoming spatial field in Ambisonics format at no additional cost.

The paper is organized as follows. In Section 2, relevant literature is reviewed and the novel aspects of the current work are outlined. Section 3 describes the notation used and reviews SH / Ambisonics definitions. In Section 4, the degenerate case of using the spherical array (with analytically-computable HRTF) as Ambisonics recording device is presented. Section 5 introduces arbitrary scatterer, outlines the procedure for computing its HRTF using numerical methods, and provides the theoretical formulation for "removal-of-the-scatterer" procedure of computing the incident field as it would be were the scatterer not present. Section 6 describes the results of simulated and real experiments both with spherical and arbitrary-shaped scatterer. Section 7 concludes the paper.

2. BACKGROUND AND RELATED WORK

In order to extract spatial information about the acoustic field, one needs to use a microphone array [1]; the physical configuration of

---

*All authors are also with Institute for Advanced Computer Studies, University of Maryland, College Park, MD 20742.

such an array obviously influences capture and processing capabilities. Said captured spatial information can be used then to reproduce the field to the listener to create spatial envelopment impression [2]. In particular, a specific spatial audio format invented simultaneously by two authors in 1972 [3] [4] for the purposes of extending then-common (and still now-common) stereo audio reproduction to third dimension (height) represents the audio field in terms of basis functions called real spherical harmonics [5] [6]; this format is known as Ambisonics [7] [8]. A specific microphone array configuration well-suited for recording data in Ambisonics format is a spherical array [9] [10], as it is naturally suited for decomposing the acoustic scene over the SH basis. While a literature suggestive of creating an Ambisonics output using spherical microphone array exists [11] [12] [13], the details of processing are mostly skimped on, perhaps because the commercial arrays used in literature are bundled with software converting raw recording to Ambisonics. This is also noted in the review [14], where methods of 3D audio production mentioned are i) use of a Soundfield microphone[1] for real scenes or ii) implementation of 3D panner for synthetic ones. The work coming closest to providing an actual, workable implementation of Ambisonics production using spherical arrays is [15]; still, even there only the standard SH decomposition equations are provided. Meanwhile, a number of practical details important to actual implementation are not covered, and the purpose of the current work is to fill those blanks in regard to the simple spherical array.

With respect to an arbitrary-shaped scatterer, the HRTF computation using a mesh representation of the body has been a subject of work for a while by different authors [16] [17]. The authors of the current work had explored fast multipole method for computing HRTF using SH basis earlier in [18] and since then have improved the computational speed by several orders of magnitude compared with existing work [19]. While traditional methods of sound field recovery operate in plane-wave (PW) basis [20] and their output can be converted into SH domain using Gegenbauer expansion [21], in the current work the SH framework is adopted throughout; this is especially convenient as the immediate output of BEM-based HRTF computation is the HRTF in SH sense[2]. Hence, to the best knowledge of authors, the presented work is the first attempt to convert the field measured at the microphones mounted on arbitrary scatterer to Ambisonics output in one step, assuming scatterer's SH HRTF is pre-computed (using BEM [18] or otherwise [22]) or measured [23].

3. DEFINITIONS

An arbitrary acoustic field $\Psi(k, \boldsymbol{r})$ in a spatial domain of radius d that does not contain acoustic sources can be decomposed over SH basis as

$$\Psi(k, \boldsymbol{r}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} C_n^m(k) j_n(kr) Y_n^m(\theta, \psi), \tag{1}$$

where k is the wavenumber, $\boldsymbol{r}$ is the three-dimensional radius-vector with components (ρ, θ, ψ), $j_n(kr)$ is the spherical Bessel function of order n, and $Y_n^m(\theta, \psi)$ are the orthonormal complex spherical

Fig. 1. The BEM model used in simulation (V = 17876, F = 35748).

harmonics defined as [24]

$$Y_n^m(\theta, \psi) = (-1)^m \sqrt{\frac{2n+1}{4\pi} \frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(\cos\theta) e^{im\psi}. \tag{2}$$

n and m are SH parameters commonly called *degree* and *order*, and $P_n^{|m|}(\mu)$ are the associated Legendre functions. Standard spherical coordinate system is used here in definition of $\boldsymbol{r}$ so that θ is a polar angle a.k.a. colatitude (0 at zenith and π at nadir) and ψ is azimuthal angle increasing clockwise. In practice, the outer summation in Eq. (1) is truncated to contain p terms. Detailed truncation error analysis is provided in [24], and the setting $p \approx (ekd-1)/2$ has been shown to provide negligible truncation error.

Ambisonics representation ignores the dependence on wavenumber and uses a different set of basis functions – *real* spherical harmonics. Shown below is the orthonormal version (called N3D normalization in the literature):

$$\tilde{Y}_n^m(\theta, \psi) = \delta_m \sqrt{2n+1} \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(\cos\theta) \Upsilon_m(\psi), \tag{3}$$

where $\Upsilon_m(\psi) = \cos(m\psi)$ when $m \geq 0$, $\sin(m\psi)$ otherwise; and δ_m is 1 when $m = 0$, $sqrt(2.0)$ otherwise. In SN3D normalization, the factor of $\sqrt{2n+1}$ is omitted. Care should be taken when comparing and implementing expressions, as symbols, angles, and normalization are defined differently in work of different authors. In particular, in Ambisonics community alternative definitions of elevation $\hat{\theta}$ and azimuth $\hat{\psi}$ are commonly used, which relate to θ and ψ used in the current work as $\theta = \pi/2 - \hat{\theta}$, $\psi = -\hat{\psi}$.

For a fixed wavenumber k, truncated Eq. (1) can be re-written in terms of real SH as

$$\Psi(k, \boldsymbol{r}) = \sum_{n=0}^{p-1} \sum_{m=-n}^{n} \tilde{C}_n^m(k) \tilde{Y}_n^m(\theta, \psi), \tag{4}$$

using a different set of expansion coefficients $\tilde{C}_n^m(k)$ and folding constant factor of $j_n(kr)$ into those coefficients (as we are interested only in angular dependence of the incident field). Note that $\tilde{C}_n^m(k)$ set is, in fact, an Ambisonics representation of the field, albeit in the frequency domain. Hence, recording a field in Ambisonics format amounts to determination of $\tilde{C}_n^m(k)$. The number $p - 1$ is called *order* of Ambisonics recording (even though technically it refers to the maximum *degree* of the SH used). The initial work [7] used $p = 2$

---

[1]A Soundfield microphone, by its principles of mechanical and signal processing design, captures the real SH of order 0 and 1.

[2]It is straightforward to convert SH HRTF to PW HRTF and vice-versa, but avoidance of unnecessary back-and-forth conversion, which can introduce inaccuracies, is important; in addition, any practical implementation requires writing appropriate software, and simpler software is faster to implement and easier to debug.
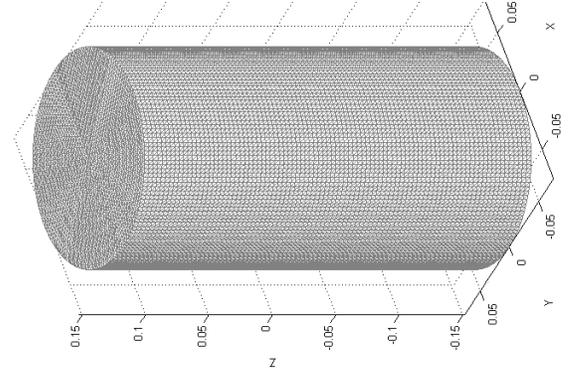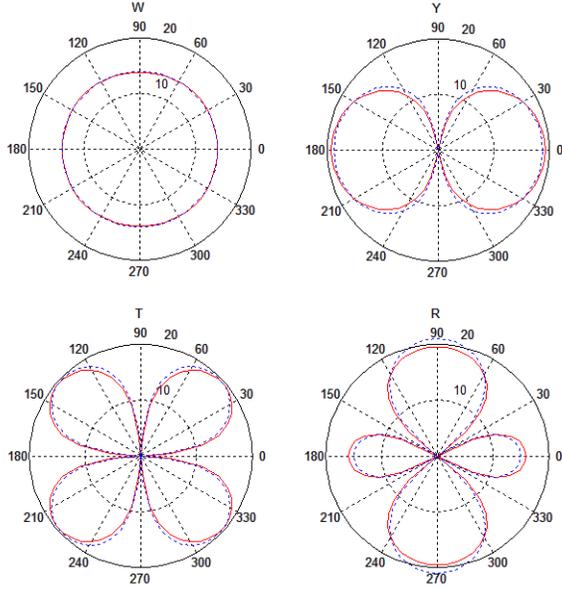
Fig. 2. Angular response magnitude for W, Y, T, R Ambisonic channels at 1.5 kHz with additive white Gaussian noise and SNR = 20 dB (solid: array response, dashed: corresponding spherical harmonic). Channel names shown are as introduced in [31] and correspond to specific SH with $n = 0, m = 0$; $n = 1, m = -1$; $n = 2, m = -1$; and $n = 2, m = 0$, respectively.

(first-order); since then, higher-order Ambisonics (HOA) techniques has been developed for p as high as 8.

The following relationship, up to a constant factor, can be trivially derived between $C_n^m(k)$ and $\tilde{C}_n^m(k)$:

$$\tilde{C}_n^0 = C_n^0, \qquad \tilde{C}_n^{-m} = i\frac{\sqrt{2}}{2}(C_n^m - C_n^{-m}),$$
$$\tilde{C}_n^m = \frac{\sqrt{2}}{2}(C_n^m + C_n^{-m}). \tag{5}$$

The rest of this paper is focused solely on computing $C_n^m(k)$ (i.e. obtaining a representation of the field in terms of traditional, complex spherical harmonics), with the conversion to $\tilde{C}_n^m(k)$ done as the final step as per above.

4. SPHERICAL ARRAY

Direct Approach: For a continuous pressure-sensitive surface of radius a, the computation of $C_n^m(k)$ is performed as [24]

$$C_n^m(k) = -i(ka)^2 i^n h_n'(ka) \int_{S_u} \Psi(k, \boldsymbol{s}) Y_n^{-m}(\boldsymbol{s}) dS(\boldsymbol{s}), \tag{6}$$

where integration is done over the sphere surface, $h_n(kr)$ is the spherical Hankel function of order n, and $\Psi(k, \boldsymbol{s})$ is the Fourier transform of the acoustic pressure at point $\boldsymbol{s}$, which is proportional to the velocity potential and is loosely referred to as the potential in this paper. Assume that L microphones are mounted on the sphere surface at points $\boldsymbol{r}_j$, $j = 1 \ldots L$. The integration can be replaced by
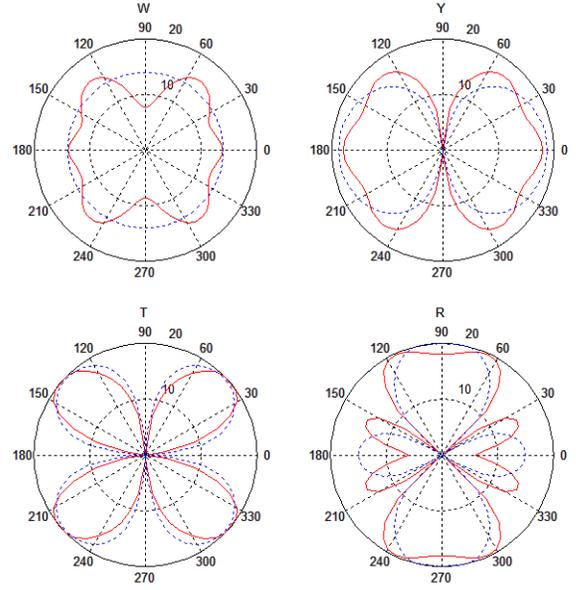

Fig. 3. Same as in Figure 2, except frequency = 3 kHz.

summation with quadrature weights ω_j:

$$C_n^m(k) = -i(ka)^2 i^n h_n'(ka) \sum_{j=1}^{L} \omega_j \Psi(k, \boldsymbol{r}_j) Y_n^{-m}(\boldsymbol{r}_j) \tag{7}$$

Least-Squares Solution: The direct approach, above, requires one to have high-quality quadrature over the sphere. An alternative approach is to figure out the potential $\Psi(k, \boldsymbol{r}_j)$ that would be created by a field described by a set of $C_n^m(k)$:

$$\Psi(k, \boldsymbol{r}_j) = 4\pi i^{-n} \frac{i}{(ka)^2} \sum_{n=0}^{p-1} \sum_{m=-n}^{n} \frac{C_n^m(k) Y_n^m(\boldsymbol{r}_j)}{h_n'(ka)}. \tag{8}$$

This equation links the mode strength and the microphone potential. The kernel

$$H_n^m(k, \boldsymbol{r}_j) = 4\pi i^{-n} \frac{i}{(ka)^2} \frac{Y_n^m(\boldsymbol{r}_j)}{h_n'(ka)} \tag{9}$$

is nothing but the SH-HRTF for the sphere, describing the potential evoked at a microphone located at $\boldsymbol{r}_j$ by a unit-strength spherical mode of degree n and order m. Given a set of measured $\Psi(k, \boldsymbol{r}_j)$ at L locations and assuming an overdetermined system (i.e. $p^2 < L$), one could compute the set of $C_n^m(k)$ that "best-fits" the observations using least-squares by multiplying measured potentials by pseudoinverse of matrix H. Even though quadrature is no longer explicitly involved, sufficiently uniform microphone distribution over the sphere is required for matrix H to be well-conditioned [25].

Practical Limitation: Given a truncation number p, the minimum number of microphones required to accurately sample the field is p^2 [24]; hence, a 64-microphone sphere can be used to record Ambisonics audio of order 7. Further limits on both lowest and highest operational frequency are imposed by physical array size and inter-microphone distance, respectively; these had been published in prior work [26] and won't be discussed further here.

5. ARBITRARY SCATTERER

Using numerical methods, it is possible to compute SH-HRTF for an *arbitrary-shaped* body; a detailed description of the fast multipole-accelerated boundary element method (BEM) involved is presented in [18, 19]. The result of the computations is the set of SH-HRTF $H_n^m(k, \boldsymbol{r})$ for arbitrary point $\boldsymbol{r}$. Assume that, via BEM computations or otherwise (e.g., via experimental measurements), SH-HRTF is known for the microphone locations $\boldsymbol{r}_j$, $j = 1 \ldots L$. The plane-wave (regular) HRTF $H(k, \boldsymbol{s}, \boldsymbol{r}_j)$ describing a potential evoked at microphone located at $\boldsymbol{r}_j$ by a plane wave arriving from direction $\boldsymbol{s}$ is expanded via SH-HRTF as

$$H(k, \boldsymbol{s}, \boldsymbol{r}_j) = \sum_{n=0}^{p-1} \sum_{m=-n}^{n} H_n^m(k, \boldsymbol{r}_j) Y_n^m(\boldsymbol{s}). \quad (10)$$

At the same time, the measured field $\Psi(k, \boldsymbol{r}_j)$ can be expanded over plane wave basis as

$$\Psi(k, \boldsymbol{r}_j) = \int_{S_u} \mu(k, \boldsymbol{s}) H(k, \boldsymbol{s}, \boldsymbol{r}_j) dS(\boldsymbol{s}), \quad (11)$$

where $\mu(k, \boldsymbol{s})$ is known as the signature function as it describes the plane wave strength as a (continuous) function of direction over the unit sphere. By further expanding it over spherical harmonics as

$$\mu(k, \boldsymbol{s}) = \sum_{n=0}^{p-1} \sum_{m=-n}^{n} C_n^m(k) Y_n^m(\boldsymbol{s}), \quad (12)$$

the problem of determining a set of $C_n^m(k)$ from the measurements $\Psi(k, \boldsymbol{r}_j)$ is reduced to solving a system of linear equations

$$\sum_{n=0}^{p-1} \sum_{m=-n}^{n} C_n^m(k) H_n^{-m}(k, \boldsymbol{r}_j) = \Psi(k, \boldsymbol{r}_j), \quad j = 1 \ldots L, \quad (13)$$

for p^2 values $C_n^m(k)$, which follows from Eq. (11) and orthonormality of spherical harmonics. When $p^2 < L$, the system is overdetermined and is solved in the least squares sense, as for sphere case. Other norms may be used in the minimization. Note that the solution above can also be derived from the sphere case (Eq. (8)) by literally replacing the sphere SH-HRTF (Eq. (9)) with BEM-computed arbitrary scatterer SH-HRTF in the equations.

6. EXPERIMENTAL RESULTS

Spherical Array: An informal experimental evaluation was performed using a 64-microphone spherical array with microphones arranged in 64-point Fliege grid [27] introduced into array processing in [28]. The microphone signals are subject to FFT to obtain $\Psi(k, \boldsymbol{r})$ for a discrete set of k. Eq. (13) is then used to obtain $C_n^m(k)$, and the IFFT is applied to $C_n^m(k)$ for each n / m combination to form the corresponding time-domain output Ambisonics signals. The resultant TOA (third-order Ambisonics) recordings were evaluated aurally using Google Jump Inspector [**?**]. Higher-order outputs (up to order seven, $p = 8$) were also created and evaluated using an internally-developed head-tracked player. Good externalization and consistent direction perception were reported by users.

Arbitrary Shape: In addition, simulated experiments were performed using a 3-inch radius, 12-inch long sound-hard cylinder as a scatterer; despite the shape being seemingly simple, analytical solution here is possible only for the case of infinite length [30]. The cylinder surface was discretized with at least 6 mesh elements per
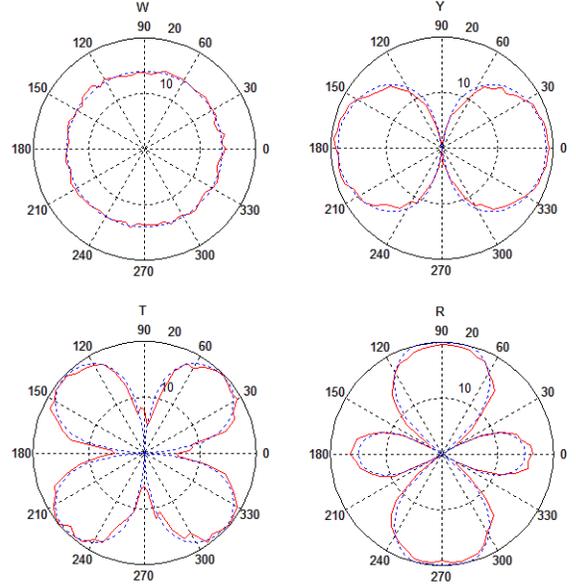

Fig. 4. Same as in Figure 2, except SNR = 0 dB.

wavelength for the highest frequency of interest (12 kHz). BEM computations were performed to compute the SH-HRTF for 16 frequencies from 0.375 to 6 kHz with a step of 375 Hz. Simulated microphones were placed on the cylinder body in 5 equispaced rings along the cylinder length with 6 equispaced microphones on each ring. In addition, the top and bottom surfaces also had 6 microphones mounted on each in a circle with a diameter of 10/3 inches, for a grand total of 42 microphones. The mesh used is shown in Figure 1. Per spatial Nyquist criteria, the aliasing frequency for the setup is approximately 2.2 kHz. Other (more complex) shapes were evaluated as well but are not reported here for the lack of space.

To evaluate accuracy of reconstructing Ambisonics signal, simulated plane-waves with additive Gaussian noise were projected on the scatterer from a number of directions. Figure 2 shows the response for the low-noise condition at a frequency of 1.5 kHz for the source orbiting the array in $X = 0$ plane in $5°$ steps. The polar response for each TOA channel matches the corresponding spherical harmonic very well; for the lack of space, only four channels are shown. Figure 3 demonstrates the deterioration of the response due to spatial aliasing at the frequency of 3 kHz. Finally, Figure 4 shows the robustness to noise; in this figure, frequency is 1.5 kHz and SNR = 0 dB. The response pattern deviates from the ideal one somewhat, but its features (lobes and nulls) are kept intact.

7. CONCLUSION

In this work, an approach to recovery of the incident acoustic field using a microphone array mounted on an arbitrarily-shaped scatterer is presented. The scatterer influence on the field is characterized through an HRTF-like transfer function, which is computed in spherical harmonics domain using numerical methods, enabling one to obtain spherical spectra of the incident field from the microphone potentials directly via least-squares fitting. Incidentally, said spherical spectra comprise Ambisonics representation of the field, allowing for use of such array as a HOA recording device. Simulations performed verify the proposed approach and show robustness to noise.

8. REFERENCES

[1] M. Brandstein and D. Ward (Eds.) (2001). "Microphone Arrays: Signal Processing Techniques and Applications", Springer-Verlag, Berlin, Germany.

[2] R. Duraiswami, D. N. Zotkin, Z. Li, E. Grassi, N. A. Gumerov, and L. S. Davis (2005). "High order spatial audio capture and its binaural head-tracked playback over headphones with HRTF cues", Proc. AES 119th Convention, New York, NY, October 2005, preprint #6540.

[3] J. J. Gibson, R. M. Christenses, and A. L. R. Limberg (1972). "Compatible FM broadcasting of panoramic sound", Journal Audio Engineering Society, vol. 20, pp. 816-822.

[4] M. A. Gerzon (1973). "Periphony: With-height sound reproduction", Journal Audio Engineering Society, vol. 21, pp. 2-10.

[5] M. A. Poletti (2005). "Three-dimensional surround sound systems based on spherical harmonics", Journal of the Audio Engineering Society, vol. 53(11), pp. 1004-1025.

[6] P. Lecomte, P.-A. Gauthier, C. Langrenne, A. Garcia, and A. Berry (2015). "On the use of a Lebedev grid for Ambisonics", Proc. AES 139th Convention, New York, NY, October 2015, preprint #9433.

[7] M. A. Gerzon (1980). "Practical periphony", Proc. AES 65th Convention, London, UK, February 1980, preprint #1571.

[8] M. A. Poletti (2000). "A unified theory of horizontal holographic sound systems", Journal of the Audio Engineering Society, vol. 48(12), pp. 1155-1182.

[9] T. Abhayapala and D. Ward (2002). "Theory and design of high order sound field microphones using spherical microphone array", Proc. IEEE ICASSP 2002, Orlando, FL, vol. 2, pp. 1949-1952.

[10] J. Meyer and G. Elko (2002). "A highly scalable spherical microphone array based on an orthonormal decomposition of the soundfield", Proc. IEEE ICASSP 2002, Orlando, FL, vol. 2, pp. 1781-1784.

[11] A. Avni and B. Rafaely (2010). "Sound localization in a sound field represented by spherical harmonics", Proc. 2nd International Symposium on Ambisonics and Spherical Acoustics, Paris, France, May 2010, pp. 1-5.

[12] S. Braun and M. Frank (2011). "Localization of 3D Ambisonic recordings and Ambisonic virtual sources", Proc. 1st International Conference on Spatial Audio (ICSA) 2011, Detmold, Germany, November 2011.

[13] S. Bertet, J. Daniel, E. Parizet, and O. Warusfel (2013). "Investigation on localisation accuracy for first and higher order Ambisonics reproduced sound sources", Acta Acustica united with Acustica, vol. 99, pp. 642-657.

[14] M. Frank, F. Zotter, and A. Sontacchi (2015). "Producing 3D audio in Ambisonics", Proc. AES 57th Intl. Conference, Hollywood, CA, March 2015, paper #14.

[15] L. Kumar (2015). "Microphone array processing for acoustic source localization in spatial and spherical harmonics domain", Ph.D. thesis, Department of Electrical Engineering, IIT Kanpur, Kanpur, Uttar Pradesh, India.

[16] Y. Tao, A. I. Tew, and S. J. Porter (2003). "The differential pressure synthesis method for efficient acoustic pressure estimation", Journal of the Audio Engineering Society, vol. 41, pp. 647-656.

[17] M. Otani and S. Ise (2006). "Fast calculation system specialized for head-related transfer function based on boundary element method", Journal of the Acoustical Society of America, vol. 119, pp. 2589-2598.

[18] N. A. Gumerov, A. E. O'Donovan, R. Duraiswami, and D. N. Zotkin (2010). "Computation of the head-related transfer function via the fast multipole accelerated boundary element method and its spherical harmonic representation", Journal of the Acoustical Society of America, vol. 127(1), pp. 370-386.

[19] N. A. Gumerov, R. Adelman, and R. Durasiwami (2013). "Fast multipole accelerated indirect boundary elements for the Helmholtz equation", Proceedings of Meetings on Acoustics, vol. 19, EID #015097.

[20] D. N. Zotkin and R. Duraiswami (2009). "Plane-wave decomposition of acoustical scenes via spherical and cylindrical microphone arrays", IEEE Transactions on Audio, Speech, and Language Processing, vol. 18(1), pp. 2-16.

[21] M. Abramowitz and I. Stegun (1964). "Handbook of Mathematical Functions", National Bureau of Standards, Government Printing Office.

[22] T. Xiao and Q.-H. Lui (2003). "Finite difference computation of head-related transfer function for human hearing", Journal of the Acoustical Society of America, vol. 113, pp. 2434-2441.

[23] D. N. Zotkin, R. Duraiswami, E. Grassi, and N. A. Gumerov (2006). "Fast head-related transfer function measurement via reciprocity", Journal of the Acoustical Society of America, vol. 120(4), pp. 2202-2215.

[24] N. A. Gumerov and R. Duraiswami (2004). "Fast multipole methods for the Helmholtz equation in three dimensions", Elsevier Science, The Netherlands.

[25] D. N. Zotkin, R. Duraiswami, and N. A. Gumerov (2009). "Regularized HRTF fitting using spherical harmonics", Proc. IEEE WASPAA 2009, New Paltz, NY, October 2009, pp. 257-260.

[26] B. Rafaely (2005). "Analysis and design of spherical microphone arrays", IEEE Transactions on Speech and Audio Processing, vol. 13(1), pp. 135-143.

[27] J. Fliege and U. Maier (1999). "The distribution of points on the sphere and corresponding cubature formulae", IMA Journal of Numerical Analysis, vol. 19, pp. 317-334.

[28] Z. Li and R. Duraiswami (2007). "Flexible and optimal design of spherical microphone arrays for beamforming", IEEE Transactions on Speech and Audio Processing, vol. 15(2), pp. 702-714.

[29] "Jump Help" (2016), retrieved from http://g.co/jump/inspector on Jan 9, 2017.

[30] H. Teutsch (2007). "Modal Array Signal Processing", Springer-Verlag, Berlin / Heidelberg, Germany.

[31] D. G. Malham (1999). "Higher order Ambisonic systems for the spatialisation of sound", Proc. ICMC 1999, Beijing, China, October 1999, pp. 484-487.