Solution to Exercise 26.1-6

The flow sum $f_1 + f_2$ satisfies skew symmetry and flow conservation, but might violate the capacity constraint.

We give proofs for skew symmetry and flow conservation and an example that shows a violation of the capacity constraint. Let $f(u,v) = (f_1 + f_2)(u,v)$.

For skew symmetry:

$$\begin{aligned} f(u,v) &= f_1(u,v) + f_2(u,v) \\ &= -f_1(v,u) - f_2(v,u) \qquad \text{(skew symmetry)} \\ &= -(f_1(v,u) + f_2(v,u)) \\ &= -f(v,u). \end{aligned}$$

For flow conservation, let $u \in V - \{s, t\}$:

$$\begin{aligned} \sum_{v \in V} f(u,v) = \sum_{v \in V} (f_1(u,v) + f_2(u,v)) &= \sum_{v \in V} f_1(u,v) + \sum_{v \in V} f_2(u,v) \\ &= 0 + 0 \qquad \text{(flow conservation)} \\ &= 0. \end{aligned}$$

For the capacity constraint, let $V = \{s, t\}$, $E = \{(s, t)\}$, and $c(s, t) = 1$. Let $f_1(s,t) = f_2(s,t) = 1$. Then f_1 and f_2 obey the capacity constraint, but $(f_1 + f_2)(u,v) = 2$, which violates the capacity constraint.

Solution to Exercise 26.1-7

To see that the flows form a convex set, we show that if f_1 and f_2 are flows, then so is $\alpha f_1 + (1-\alpha) f_2$ for all α such that $0 \le \alpha \le 1$.

For the capacity constraint, first observe that $\alpha \le 1$ implies that $1-\alpha \ge 0$. Thus, for any $u,\ v \in V$, we have

$$\begin{aligned} \alpha f_1(u,v) + (1-\alpha) f_2(u,v) &\ge 0 \cdot f_1(u,v) + 0 \cdot f_2(u,v) \\ &= 0. \end{aligned}$$

Since $f_1(u,v) \le c(u,v)$ and $f_2(u,v) \le c(u,v)$, we also have

$$\begin{aligned} \alpha f_1(u,v) + (1-\alpha) f_2(u,v) &\le \alpha \cdot c(u,v) + (1-\alpha) \cdot c(u,v) \\ &= (\alpha + (1-\alpha)) c(u,v) \\ &= c(u,v). \end{aligned}$$

For skew symmetry, we have $f_1(u,v) = -f_1(v,u)$ and $f_2(u,v) = -f_2(v,u)$ for any $u, v \in V$. Thus, we have

$$\alpha f_1(u,v) + (1-\alpha) f_2(u,v) = -\alpha f_1(v,u) - (1-\alpha) f_2(v,u)$$
$$= -(\alpha f_1(v,u) + (1-\alpha) f_2(v,u)).$$

For flow conservation, observe that since f_1 and f_2 obey flow conservation, we Have $\sum_{v \in V} f_1(u,v) = 0$ and $\sum_{v \in V} f_2(u,v) = 0$ for any $u \in V - \{s, t\}$. Thus,

$$\sum_{v \in V} (\alpha f_1(u,v) + (1-\alpha) f_2(u,v)) = \alpha \sum_{v \in V} f_1(u,v) + (1-\alpha) \sum_{v \in V} f_2(u,v)$$
$$= \alpha \cdot 0 + (1-\alpha) \cdot 0$$
$$= 0.$$

Solution to Exercise 26.2-9

For any two vertices u and v in G, you can define a flow network G_{uv} consisting of the directed version of G with all edge capacities set to 1, $s = u$, and $t = v$. (G_{uv} has $O(V)$ vertices—actually, $|V|$—and $O(E)$ edges, as required. We want all capacities to be 1 so that the number of edges crossing a cut equals the capacity of the cut.) Let f_{uv} denote a maximum flow in G_{uv}.

We claim that for any $u \in V$, the edge connectivity k equals $\min_{v \in V - \{u\}} |f_{uv}|$. We'll show below that this claim holds. Assuming that it holds, we can find k as follows:

EDGE-CONNECTIVITY(G)

select any vertex $u \in V$

for each vertex $v \in V - \{u\}$ ▷ $|V| - 1$ iterations

  do set up the flow network G_{uv} as described above

    find the maximum flow f_{uv} on G_{uv}

return the minimum of the $|V| - 1$ max-flow values: $\min_{v \in V - \{u\}} |f_{uv}|$

The claim follows from the max-flow min-cut theorem and how we chose capacities so that the capacity of a cut is the number of edges crossing it. We prove that $k = \min_{v \in V - \{u\}} |f_{uv}|$, for any $u \in V$ by showing separately that k is at least this minimum and that k is at most this minimum.

- Proof that $k \geq \min_{v \in V - \{u\}} |f_{uv}|$:

Let $m = \min_{v \in V - \{u\}} |f_{uv}|$. Suppose we remove only $m - 1$ edges from G. For any vertex v, by the max-flow min-cut theorem, u and v are still connected. (The max flow from u to v is at least m, hence any cut separating u from v has capacity at least m, which means at least m edges cross any such cut. Thus at least one edge is left crossing the cut when we remove $m-1$ edges.) Thus every node is connected to u, which implies that the graph is still connected. So at least m edges must be removed to disconnect the graph—i.e., $k \geq \min_{v \in V-\{u\}} |f_{uv}|$.

- Proof that $k \leq \min_{v \in V-\{u\}} |f_{uv}|$:

Consider a vertex v with the minimum $|f_{uv}|$. By the max-flow min-cut theorem, there is a cut of capacity $|f_{uv}|$ separating u and v. Since all edge capacities are 1, exactly $|f_{uv}|$ edges cross this cut. If these edges are removed, there is no path from u to v, and so our graph becomes disconnected. Hence $k \leq \min_{v \in V-\{u\}} |f_{uv}|$.

- Thus, the claim that $k = \min_{v \in V-\{u\}} |f_{uv}|$, for any $u \in V$ is true.

Solution to Problem 26-4

a. Just execute one iteration of the Ford-Fulkerson algorithm. The edge (u, v) in E with increased capacity ensures that the edge (u, v) is in the residual graph. So look for an augmenting path and update the flow if a path is found.

Time: $O(V + E) = O(E)$ if we find the augmenting path with either depth-first or breadth-first search.

To see that only one iteration is needed, consider separately the cases in which (u, v) is or is not an edge that crosses a minimum cut. If (u, v) does not cross a minimum cut, then increasing its capacity does not change the capacity of any minimum cut, and hence the value of the maximum flow does not change. If (u, v) does cross a minimum cut, then increasing its capacity by 1 increases the capacity of that minimum cut by 1, and hence possibly the value of the maximum flow by 1. In this case, there is either no augmenting path (in which case there was some other minimum cut that (u, v) does not cross), or the augmenting path increases flow by 1. No matter what, one iteration of Ford-Fulkerson suffices.

b. Let f be the maximum flow before reducing $c(u, v)$.

If $f(u, v) = 0$, we don't need to do anything.

If $f(u, v) > 0$, we will need to update the maximum flow. Assume from now on that $f(u, v) > 0$, which in turn implies that $f(u, v) \geq 1$.

Define $f'(x, y) = f(x, y)$ for all $x, y \in V$, except that $f'(u, v) = f(u, v) - 1$. Although f' obeys all capacity contraints, even after $c(u, v)$ has been reduced, it is not a legal flow, as it violates skew symmetry and flow conservation at u and v. f' has one more unit of flow entering u than leaving u, and it has one more unit of flow leaving v than entering v.

The idea is to try to reroute this unit of flow so that it goes out of u and into v via some other path. If that is not possible, we must reduce the flow from s to u and from v to t by one unit.

Look for an augmenting path from u to v (note: *not* from s to t).

- If there is such a path, augment the flow along that path.
- If there is no such path, reduce the flow from s to u by augmenting the flow from u to s. That is, find an augmenting path $u \leadsto s$ and augment the flow along that path. (There definitely is such a path, because there is flow from s to u.) Similarly, reduce the flow from v to t by finding an augmenting path $t \leadsto v$ and augmenting the flow along that path.

Time: $O(V + E) = O(E)$ if we find the paths with either DFS or BFS.