ENEE 324

Solution HW #13

Problem 5.1.3 Solution

(a) In terms of the joint PDF, we can write joint CDF as

$$F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f_{X_1,\ldots,X_n}(y_1,\ldots,y_n)\, dy_1 \cdots dy_n \tag{1}$$

However, simplifying the above integral depends on the values of each x_i. In particular, $f_{X_1,\ldots,X_n}(y_1,\ldots,y_n) = 1$ if and only if $0 \le y_i \le 1$ for each i. Since $F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = 0$ if any $x_i < 0$, we limit, for the moment, our attention to the case where $x_i \ge 0$ for all i. In this case, some thought will show that we can write the limits in the following way:

$$\begin{aligned} F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) &= \int_0^{\max(1,x_1)} \cdots \int_0^{\min(1,x_n)} dy_1 \cdots dy_n && (2)\\ &= \min(1,x_1)\min(1,x_2)\cdots\min(1,x_n) && (3) \end{aligned}$$

A complete expression for the CDF of $X_1,\ldots,X_n$ is

$$F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \begin{cases} \prod_{i=1}^{n}\min(1,x_i) & 0 \le x_i, i = 1,2,\ldots,n \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(b) For $n = 3$,

$$\begin{aligned} 1 - P\left[\min_i X_i \le 3/4\right] &= P\left[\min_i X_i > 3/4\right] && (5)\\ &= P\left[X_1 > 3/4, X_2 > 3/4, X_3 > 3/4\right] && (6)\\ &= \int_{3/4}^{1}\int_{3/4}^{1}\int_{3/4}^{1} dx_1\, dx_2\, dx_3 && (7)\\ &= (1-3/4)^3 = 1/64 && (8) \end{aligned}$$

Thus $P[\min_i X_i \le 3/4] = 63/64$.

Problem 5.3.5 Solution

The value of each byte is an independent experiment with 255 possible outcomes. Each byte takes on the value b_i with probability $p_i = p = 1/255$. The joint PMF of $N_0, \ldots, N_{255}$ is the multinomial PMF

$$P_{N_0,\ldots,N_{255}}(n_0, \ldots, n_{255}) = \frac{10000!}{n_0! n_1! \cdots n_{255}!} p^{n_0} p^{n_1} \cdots p^{n_{255}} \qquad n_0 + \cdots + n_{255} = 10000 \quad (1)$$

$$= \frac{10000!}{n_0! n_1! \cdots n_{255}!} (1/255)^{10000} \qquad n_0 + \cdots + n_{255} = 10000 \quad (2)$$

To evaluate the joint PMF of N_0 and N_1, we define a new experiment with three categories: b_0, b_1 and "other." Let $\hat{N}$ denote the number of bytes that are "other." In this case, a byte is in the "other" category with probability $\hat{p} = 253/255$. The joint PMF of N_0, N_1, and $\hat{N}$ is

$$P_{N_0,N_1,\hat{N}}(n_0, n_1, \hat{n}) = \frac{10000!}{n_0! n_1! \hat{n}!} \left(\frac{1}{255}\right)^{n_0} \left(\frac{1}{255}\right)^{n_1} \left(\frac{253}{255}\right)^{\hat{n}} \qquad n_0 + n_1 + \hat{n} = 10000 \quad (3)$$

Now we note that the following events are one in the same:

$$\{N_0 = n_0, N_1 = n_1\} = \left\{N_0 = n_0, N_1 = n_1, \hat{N} = 10000 - n_0 - n_1\right\} \quad (4)$$

Hence, for non-negative integers n_0 and n_1 satisfying $n_0 + n_1 \leq 10000$,

$$P_{N_0,N_1}(n_0, n_1) = P_{N_0,N_1,\hat{N}}(n_0, n_1, 10000 - n_0 - n_1) \quad (5)$$

$$= \frac{10000!}{n_0! n_1! (10000 - n_0 - n_1)!} \left(\frac{1}{255}\right)^{n_0+n_1} \left(\frac{253}{255}\right)^{10000-n_0-n_1} \quad (6)$$

Note : p_i shoud be equal to 1/256 in the above soution since there are 256 possible outcome not 255.

Problem 5.4.6 Solution

We find the marginal PDFs using Theorem 5.5. First we note that for $x < 0$, $f_{X_i}(x) = 0$. For $x_1 \geq 0$,

$$f_{X_1}(x_1) = \int_{x_1}^{\infty} \left(\int_{x_2}^{\infty} e^{-x_3}\, dx_3 \right) dx_2 = \int_{x_1}^{\infty} e^{-x_2}\, dx_2 = e^{-x_1} \tag{1}$$

Similarly, for $x_2 \geq 0$, X_2 has marginal PDF

$$f_{X_2}(x_2) = \int_{0}^{x_2} \left(\int_{x_2}^{\infty} e^{-x_3}\, dx_3 \right) dx_1 = \int_{0}^{x_2} e^{-x_2}\, dx_1 = x_2 e^{-x_2} \tag{2}$$

Lastly,

$$f_{X_3}(x_3) = \int_{0}^{x_3} \left(\int_{x_1}^{x_3} e^{-x_3}\, dx_2 \right) dx_1 = \int_{0}^{x_3} (x_3 - x_1) e^{-x_3}\, dx_1 \tag{3}$$

$$= -\frac{1}{2}(x_3 - x_1)^2 e^{-x_3} \Big|_{x_1=0}^{x_1=x_3} = \frac{1}{2} x_3^2 e^{-x_3} \tag{4}$$

The complete expressions for the three marginal PDFs are

$$f_{X_1}(x_1) = \begin{cases} e^{-x_1} & x_1 \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

$$f_{X_2}(x_2) = \begin{cases} x_2 e^{-x_2} & x_2 \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

$$f_{X_3}(x_3) = \begin{cases} (1/2) x_3^2 e^{-x_3} & x_3 \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

In fact, each X_i is an Erlang $(n, \lambda) = (i, 1)$ random variable.

Problem 5.5.3 Solution

The response time X_i of the ith truck has PDF $f_{X_i}(x_i)$ and CDF $F_{X_i}(x_i)$ given by

$$f_{X_i}(x_i) = \begin{cases} \frac{1}{2}e^{-x/2} & x \geq 0, \\ 0 & \text{otherwise,} \end{cases} \qquad F_{X_i}(x_i) = F_X(x_i) = \begin{cases} 1 - e^{-x/2} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Let $R = \max(X_1, X_2, \ldots, X_6)$ denote the maximum response time. From Theorem 5.7, R has PDF

$$F_R(r) = (F_X(r))^6. \tag{2}$$

(a) The probability that all six responses arrive within five seconds is

$$P[R \leq 5] = F_R(5) = (F_X(5))^6 = (1 - e^{-5/2})^6 = 0.5982. \tag{3}$$

(b) This question is worded in a somewhat confusing way. The "expected response time" refers to $E[X_i]$, the response time of an individual truck, rather than $E[R]$. If the expected response time of a truck is τ, then each X_i has CDF

$$F_{X_i}(x) = F_X(x) = \begin{cases} 1 - e^{-x/\tau} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

The goal of this problem is to find the maximum permissible value of τ. When each truck has expected response time τ, the CDF of R is

$$F_R(r) = (F_X(x)\, r)^6 = \begin{cases} (1 - e^{-r/\tau})^6 & r \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

We need to find τ such that

$$P[R \leq 3] = (1 - e^{-3/\tau})^6 = 0.9. \tag{6}$$

This implies

$$\tau = \frac{-3}{\ln\left(1 - (0.9)^{1/6}\right)} = 0.7406 \text{ s.} \tag{7}$$

Problem 5.6.8 Solution
The 2-dimensional random vector $\mathbf{Y}$ has PDF

$$f_{\mathbf{Y}}(\mathbf{y}) = \begin{cases} 2 & \mathbf{y} \geq \mathbf{0}, \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{y} \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Rewritten in terms of the variables y_1 and y_2,

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 2 & y_1 \geq 0, y_2 \geq 0, y_1 + y_2 \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

In this problem, the PDF is simple enough that we can compute $E[Y_i^n]$ for arbitrary integers $n \geq 0$.

$$E[Y_1^n] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y_1^n f_{Y_1,Y_2}(y_1, y_2)\, dy_1\, dy_2 = \int_0^1 \int_0^{1-y_2} 2y_1^n\, dy_1\, dy_2. \tag{3}$$

A little calculus yields

$$E[Y_1^n] = \int_0^1 \left(\frac{2}{n+1} y_1^{n+1} \bigg|_0^{1-y_2} \right) dy_2 = \frac{2}{n+1} \int_0^1 (1-y_2)^{n+1}\, dy_2 = \frac{2}{(n+1)(n+2)}. \tag{4}$$

Symmetry of the joint PDF $f_{Y_{1,2}}(y_{1,2})$ implies that $E[Y_2^n] = E[Y_1^n]$. Thus, $E[Y_1] = E[Y_2] = 1/3$ and

$$E[\mathbf{Y}] = \boldsymbol{\mu}_{\mathbf{Y}} = \begin{bmatrix} 1/3 & 1/3 \end{bmatrix}'. \tag{5}$$

In addition,

$$R_{\mathbf{Y}}(1,1) = E\left[Y_1^2\right] = 1/6, \qquad R_{\mathbf{Y}}(2,2) = E\left[Y_2^2\right] = 1/6. \tag{6}$$

To complete the correlation matrix, we find

$$R_{\mathbf{Y}}(1,2) = E[Y_1 Y_2] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y_1 y_2 f_{Y_1,Y_2}(y_1, y_2)\, dy_1\, dy_2 = \int_0^1 \int_0^{1-y_2} 2y_1 y_2\, dy_1\, dy_2. \tag{7}$$

Following through on the calculus, we obtain

$$R_{\mathbf{Y}}(1,2) = \int_0^1 \left(y_1^2 \Big|_0^{1-y-2} \right) y_2\, dy_2 = \int_0^1 y_2 (1-y_2)^2\, dy_2 = \frac{1}{2} y_2^2 - \frac{2}{3} y_2^3 + \frac{1}{4} y_2^4 \bigg|_0^1 = \frac{1}{12}. \tag{8}$$

Thus we have found that

$$\mathbf{R}_{\mathbf{Y}} = \begin{bmatrix} E\left[Y_1^2\right] & E[Y_1 Y_2] \\ E[Y_2 Y_1] & E\left[Y_2^2\right] \end{bmatrix} = \begin{bmatrix} 1/6 & 1/12 \\ 1/12 & 1/6 \end{bmatrix}. \tag{9}$$

Lastly, $\mathbf{Y}$ has covariance matrix

$$\mathbf{C}_{\mathbf{Y}} = \mathbf{R}_{\mathbf{Y}} - \boldsymbol{\mu}_{\mathbf{Y}} \boldsymbol{\mu}_{\mathbf{Y}}' = \begin{bmatrix} 1/6 & 1/12 \\ 1/12 & 1/6 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 \end{bmatrix} \tag{10}$$

$$= \begin{bmatrix} 1/9 & -1/36 \\ -1/36 & 1/9 \end{bmatrix}. \tag{11}$$