Solution for HW #8

Problem 3.4.10 Solution

The integral I_1 is

$$I_1 = \int_0^\infty \lambda e^{-\lambda x}\, dx = -e^{-\lambda x}\Big|_0^\infty = 1 \qquad (1)$$

For $n > 1$, we have

$$I_n = \int_0^\infty \underbrace{\frac{\lambda^{n-1} x^{n-1}}{(n-1)!}}_{u} \underbrace{\lambda e^{-\lambda x}\, dt}_{dv} \qquad (2)$$

We define u and dv as shown above in order to use the integration by parts formula $\int u\, dv = uv - \int v\, du$. Since

$$du = \frac{\lambda^{n-1} x^{n-1}}{(n-2)!}\, dx \qquad v = -e^{-\lambda x} \qquad (3)$$

we can write

$$I_n = uv|_0^\infty - \int_0^\infty v\, du \qquad (4)$$

$$= -\frac{\lambda^{n-1} x^{n-1}}{(n-1)!} e^{-\lambda x}\Big|_0^\infty + \int_0^\infty \frac{\lambda^{n-1} x^{n-1}}{(n-2)!} e^{-\lambda x}\, dx = 0 + I_{n-1} \qquad (5)$$

Hence, $I_n = \mathbf{1}$ for all $n \geq 1$.

Problem 3.4.13 Solution

For $n = 1$, we have the fact $E[X] = 1/\lambda$ that is given in the problem statement. Now we assume that $E[X^{n-1}] = (n-1)!/\lambda^{n-1}$. To complete the proof, we show that this implies that $E[X^n] = n!/\lambda^n$. Specifically, we write

$$E\left[X^n\right] = \int_0 x^n \lambda e^{-\lambda x}\, dx \qquad (1)$$

Now we use the integration by parts formula $\int u\, dv = uv - \int v\, du$ with $u = x^n$ and $dv = \lambda e^{-\lambda x}\, dx$. This implies $du = nx^{n-1}\, dx$ and $v = -e^{-\lambda x}$ so that

$$E\left[X^n\right] = -x^n e^{-\lambda x}\Big|_0^\infty + \int_0^\infty n x^{n-1} e^{-\lambda x}\, dx \qquad (2)$$

$$= 0 + \frac{n}{\lambda} \int_0^\infty x^{n-1} \lambda e^{-\lambda x}\, dx \qquad (3)$$

$$= \frac{n}{\lambda} E\left[X^{\mathbf{n-1}}\right] \qquad (4)$$

By our induction hyothesis, $E[X^{n-1}] = (n-1)!/\lambda^{n-1}$ which implies

$$E\left[X^n\right] = n!/\lambda^n \qquad (5)$$

Problem 3.5.7 Solution

We are given that there are 100,000,000 men in the United States and 23,000 of them are at least 7 feet tall, and the heights of U.S men are independent Gaussian random variables with mean $5'10''$.

(a) Let H denote the height in inches of a U.S male. To find σ_X, we look at the fact that the probability that $P[H \geq 84]$ is the number of men who are at least 7 feet tall divided by the total number of men (the frequency interpretation of probability). Since we measure H in inches, we have

$$P[H \geq 84] = \frac{23{,}000}{100{,}000{,}000} = \Phi\left(\frac{70-84}{\sigma_X}\right) = 0.00023 \tag{1}$$

Since $\Phi(-x) = 1 - \Phi(x) = Q(x)$,

$$Q(14/\sigma_X) = 2.3 \cdot 10^{-4} \tag{2}$$

From Table 3.2, this implies $14/\sigma_X = 3.5$ or $\sigma_X = 4$.

(b) The probability that a randomly chosen man is at least 8 feet tall is

$$P[H \geq 96] = Q\left(\frac{96-70}{4}\right) = Q(6.5) \tag{3}$$

Unfortunately, Table 3.2 doesn't include $Q(6.5)$, although it should be apparent that the probability is very small. In fact, $Q(6.5) = 4.0 \times 10^{-11}$.

(c) First we need to find the probability that a man is at least 7'6".

$$P[H \geq 90] = Q\left(\frac{90-70}{4}\right) = Q(5) \approx 3 \cdot 10^{-7} = \beta \tag{4}$$

Although Table 3.2 stops at $Q(4.99)$, if you're curious, the exact value is $Q(5) = 2.87 \cdot 10^{-7}$.

Now we can begin to find the probability that no man is at least 7'6". This can be modeled as 100,000,000 repetitions of a Bernoulli trial with parameter $1-\beta$. The probability that no man is at least 7'6" is

$$(1-\beta)^{100{,}000{,}000} = 9.4 \times 10^{-14} \tag{5}$$

(d) The expected value of N is just the number of trials multiplied by the probability that a man is at least 7'6".

$$E[N] = 100{,}000{,}000 \cdot \beta = 30 \tag{6}$$

Problem 3.5.9 Solution

First we note that since W has an $N[\mu, \sigma^2]$ distribution, the integral we wish to evaluate is

$$I = \int_{-\infty}^{\infty} f_W(w)\, dw = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-(w-\mu)^2/2\sigma^2}\, dw \tag{1}$$

(a) Using the substitution $x = (w - \mu)/\sigma$, we have $dx = dw/\sigma$ and

$$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx \tag{2}$$

(b) When we write I^2 as the product of integrals, we use y to denote the other variable of integration so that

$$I^2 = \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx\right)\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\, dy\right) \tag{3}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\, dx\, dy \tag{4}$$

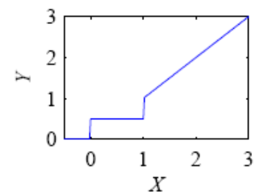
(c) By changing to polar coordinates, $x^2 + y^2 = r^2$ and $dx\, dy = r\, dr\, d\theta$ so that

$$I^2 = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} e^{-r^2/2} r\, dr\, d\theta \tag{5}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} -e^{-r^2/2}\Big|_0^{\infty}\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} d\theta = 1 \tag{6}$$

Problem 3.7.15 Solution

The relationship between X and Y is shown in the following figure:



(a) Note that $Y = 1/2$ if and only if $0 \le X \le 1$. Thus,

$$P[Y = 1/2] = P[0 \le X \le 1] = \int_0^1 f_X(x)\ dx = \int_0^1 (x/2)\, dx = 1/4 \tag{1}$$

(b) Since $Y \ge 1/2$, we can conclude that $F_Y(y) = 0$ for $y < 1/2$. Also, $F_Y(1/2) = P[Y = 1/2] = 1/4$. Similarly, for $1/2 < y \le 1$,

$$F_Y(y) = P[0 \le X \le 1] = P[Y = 1/2] = 1/4 \tag{2}$$

Next, for $1 < y \le 2$,

$$F_Y(y) = P[X \le y] = \int_0^y f_X(x)\ dx = y^2/4 \tag{3}$$

Lastly, since $Y \le 2$, $F_Y(y) = 1$ for $y \ge 2$. The complete expression of the CDF is

$$F_Y(y) = \begin{cases} 0 & y < 1/2 \\ 1/4 & 1/2 \le y \le 1 \\ y^2/4 & 1 < y < 2 \\ 1 & y \ge 2 \end{cases} \tag{4}$$

Problem 3.8.5 Solution

(a) We first find the conditional PDF of T. The PDF of T is

$$f_T(t) = \begin{cases} 100e^{-100t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The conditioning event has probability

$$P[T > 0.02] = \int_{0.02}^{\infty} f_T(t)\, dt = -e^{-100t}\Big|_{0.02}^{\infty} = e^{-2} \tag{2}$$

From Definition 3.15, the conditional PDF of T is

$$f_{T|T>0.02}(t) = \begin{cases} \frac{f_T(t)}{P[T>0.02]} & t \geq 0.02 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 100e^{-100(t-0.02)} & t \geq 0.02 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

The conditional expected value of T is

$$E[T|T > 0.02] = \int_{0.02}^{\infty} t(100)e^{-100(t-0.02)}\, dt \tag{4}$$

The substitution $\tau = t - 0.02$ yields

$$\begin{aligned} E[T|T > 0.02] &= \int_{0}^{\infty} (\tau + 0.02)(100)e^{-100\tau}\, d\tau & (5) \\ &= \int_{0}^{\infty} (\tau + 0.02) f_T(\tau)\, d\tau = E[T + 0.02] = 0.03 & (6) \end{aligned}$$

(b) The conditional second moment of T is

$$E\left[T^2|T > 0.02\right] = \int_{0.02}^{\infty} t^2(100)e^{-100(t-0.02)}\, dt \tag{7}$$

The substitution $\tau = t - 0.02$ yields

$$\begin{aligned} E\left[T^2|T > 0.02\right] &= \int_{0}^{\infty} (\tau + 0.02)^2(100)e^{-100\tau}\, d\tau & (8) \\ &= \int_{0}^{\infty} (\tau + 0.02)^2 f_T(\tau)\, d\tau & (9) \\ &= E\left[(T + 0.02)^2\right] & (10) \end{aligned}$$

Now we can calculate the conditional variance.

$$\begin{aligned} \text{Var}[T|T > 0.02] &= E\left[T^2|T > 0.02\right] - (E[T|T > 0.02])^2 & (11) \\ &= E\left[(T + 0.02)^2\right] - (E[T + 0.02])^2 & (12) \\ &= \text{Var}[T + 0.02] & (13) \\ &= \text{Var}[T] = 0.01 & (14) \end{aligned}$$