HW 15

노트 제목 2009-04-21

6.3.3 6.4.4 6.4.5 6.4.7.

6.3.3. $X \sim \text{unif}[a, b]$.

MGF $\phi_X(s)$? $E[X]$ $E[X^2]$.

$$\phi_X(s) = E[e^{sx}]$$
$$= \int_a^b e^{sx} \frac{1}{b-a} dx.$$
$$= \frac{1}{b-a} \frac{1}{s} \left[e^{sx}\right]_a^b$$
$$= \frac{1}{b-a} \frac{1}{s} \left(e^{sb} - e^{sa}\right).$$
$$= \frac{1}{s(b-a)} \left(e^{sb} - e^{sa}\right).$$

$$\left.\frac{\phi_X(s)}{ds}\right]_{s=0} = \frac{1}{b-a}\left(\frac{e^{sb}}{s} - \frac{e^{sa}}{s}\right) = \frac{1}{b-a}\left(\frac{be^{sb}s - e^{sb}}{s^2} - \frac{ae^{sa}s - e^{sa}}{s^2}\right).$$

$\frac{0}{0} \Rightarrow$ L'Hopital's rule

$$E[X] = \lim_{s \to 0} \frac{b^2 e^{bs} - a^2 e^{as}}{2(b-a)} = \frac{a+b}{2}$$

$$E[X^2] = \frac{\phi_x^2(s)}{ds^2}\Big]_{s=0} = \frac{s^2[b^2e^{bs}-a^2e^{as}]-2s[be^{bs}-ae^{as}]}{(b-a)s^3} + \frac{2[be^{bs}-ae^{s}]}{(b-a)s^3}$$

$$\frac{0}{0}$$

$$E[X^2] = \lim_{s\to 0} \frac{s^2[b^3e^{bs}-a^3e^{as}]}{3(b-a)s^2} = \frac{b^2+ab+a^2}{3}$$

6.4.4. $P[W] = \frac{1}{3}$.

win	+2
tie	+1
Loss	0

X_i: the number of points in game i.

Y: total number of points earned over n games.

(a). $\phi_{X_i}(s)$ $\phi_Y(s)$.

$$X_i \begin{cases} 2 & \text{Prob} \to \frac{1}{3} \\ 1 & \to \frac{1}{3} \\ 0 & \to \frac{1}{3} \end{cases}$$

$$\phi_{X_i}(s) = E[e^{sX}] = \sum_{x=0}^{2} e^{sx} \times \frac{1}{3} = \frac{1}{3}[1+e^s+e^{2s}].$$

$Y = X_1 + \cdots + X_n$.

$$\phi_Y(s) = [\phi_{X_i}(s)]^n = \frac{[1+e^s+e^{2s}]^n}{3^n}.$$

(b). $E[X_i] = \frac{1}{3}\times 2 + \frac{1}{3}\times 1 + \frac{1}{3}\times 0 = 1.$

$E[X_i^2] = \frac{5}{3}$

$Var[X_i] = \frac{5}{3} - (1)^2 = \frac{2}{3}.$

$E[T] = nE[X] = n.$

$Var[T] = nVar[X] = \frac{2}{3}n.$

6.4.5.

$$P_{K_i}(k) = \begin{cases} \frac{2^k e^{-2}}{k!} & k = 0, 1, 2, \cdots \\ 0 & o.w. \end{cases}$$

K_i i i.i.d.

$R_i = K_1 + K_2 + \cdots + K_i.$

(a) $\phi_{K_i}(s) = E[e^{sK_i}] = \sum_{k=0}^{\infty} e^{sk} \frac{2^k e^{-2}}{k!}$

$= e^{-2} \sum_{k=0}^{\infty} \frac{(2e^s)^k}{k!}$

$= e^{-2} \cdot e^{+2e^s} = e^{2(e^s - 1)}.$

(b). $\phi_{R_i}(s) = \prod_{n=1}^{i} \phi_K(s) = e^{2i(e^s - 1)}$

(c). $P_{R_i}(r)$.

$$E[e^{R_i s}] = \sum_{R=0}^{\infty} e^{Rs} \cdot P_R(r) = e^{2i(e^s-1)}.$$

$$= e^{-2} e^{+2ie^s}$$

$$P_{R_i}(r) = \begin{cases} (2i)^r e^{-2i}/r! & r=0,1,2,\dots \\ 0 & o.w \end{cases}$$

(d). $E[R_i] \Rightarrow 2i$ (Poisson r.v mean $2i$).

$Var[R_i] \Rightarrow 2i$ "

6.4.7. $K_1, K_2, \dots$ independent sample of r.v K.

$M = K_1 + \dots + K_n$.

(a) $E[M] = nE[K]$.

pf) $\phi_M(s) = [\phi_K(s)]^n$. $E[e^{sx}] \Rightarrow E[1] = 1$ $s=0$

$$\frac{d\phi_M(s)}{ds} = n[\phi_K(s)]^{n-1} \frac{d\phi_K(s)}{ds}$$

$$E[M] = n[\phi_K(s)]^{n-1} \times \frac{d\phi_K(s)}{ds}\Big|_{s=0} = nE[K]$$

(b). $$\frac{d^2\phi_M(s)}{ds^2} = n(n-1)\left[\phi_K(s)\right]^{n-2}\left(\frac{d\phi_K(s)}{ds}\right)^2 + n\left[\phi_K(s)\right]^{n-1}\frac{d^2\phi_K(s)}{ds^2}$$

$$E[M^2] = \frac{d^2\phi_M(s)}{ds^2}\Big|_{s=0} = n(n-1)\left(E[K]\right)^2 + nE[K^2].$$