derivations in the exercises will no doubt encounter a notoriously troublesome problem connected with proofs; namely, while it is relatively simple to verify that a given proof is correct, it may be very difficult to find the one one wants. So if presented with a problem such as (1-29), one might have to try many unsuccessful paths before finding one that leads to the desired final expression. A certain amount of cutting and trying is therefore to be expected.

Exercises

1. Given the following sets:

$$\begin{array}{lllll} A = \{a,b,c,2,3,4\} & & E = \{a,b,\{c\}\} \\ B = \{a,b\} & & F = \emptyset \\ C = \{c,2\} & & G = \{\{a,b\},\{c,2\}\} \\ D = \{b,c\} \end{array}$$

classify each of the following statements as true or false

(a)	$c \in A$	(g)	$D \subset A$	(m)	$B \subseteq G$
(b)	$c \in F$	(h)	$A \subseteq C$	(n)	$\{B\} \subseteq G$
(c)	$c \in E$	(i)	$D \subseteq E$	(o)	$D \subseteq G$
(d)	$\{c\} \in E$	(j)	$F \subseteq A$	(p)	$\{D\} \subseteq G$
(e)	$\{c\} \in C$	(k)	$E \subseteq F$	(q)	$G \subseteq A$
(f)	$B \subseteq A$	(l)	$B \in G$	(r)	$\{\{c\}\} \subseteq E$

2. For any arbitrary set S,

   (a) is S a member of $\{S\}$?

   (b) is $\{S\}$ a member of $\{S\}$?

   (c) is $\{S\}$ a subset of $\{S\}$?

   (d) what is the set whose only member is $\{S\}$?

3. Write a specification by rules and one by predicates for each of the following sets. Remember that there is no order assumed in the list, so you cannot use words like 'the first' or 'the latter'. Recall also that a recursive rule may contain more than one if-then statement.

   (a) $\{5, 10, 15, 20, \ldots\}$

   (b) $\{7, 17, 27, 37, \ldots\}$

   (c) $\{300, 301, 302, \ldots, 399, 400\}$

   (d) $\{3, 4, 7, 8, 11, 12, 15, 16, 19, 20, \ldots\}$

(e) $\{0, 2, -2, 4, -4, 6, -6, \ldots\}$

(f) $\{1, 1/2, 1/4, 1/8, 1/16, \ldots\}$

4. Consider the following sets:

$$\begin{array}{lll@{\qquad}lll} S1 & = & \{\{\emptyset\}, \{A\}, A\} & S6 & = & \emptyset \\ S2 & = & A & S7 & = & \{\emptyset\} \\ S3 & = & \{A\} & S8 & = & \{\{\emptyset\}\} \\ S4 & = & \{\{A\}\} & S9 & = & \{\emptyset, \{\emptyset\}\} \\ S5 & = & \{\{A\}, A\} & & & \end{array}$$

Answer the following questions. Remember that the members of a set are the items separated by commas, if there is more than one, between the outermost braces only; a subset is formed by enclosing within braces zero or more of the members of a given set, separated by commas.

(a) Of the sets $S1$ - $S9$ which are members of $S1$?

(b) which are subsets of $S1$?

(c) which are members of $S9$?

(d) which are subsets of $S9$?

(e) which are members of $S4$?

(f) which are subsets of $S4$?

5. Specify each of the following sets by listing its members:

(a)	$\wp\{a, b, c\}$	(d)	$\wp\{\emptyset\}$
(b)	$\wp\{a\}$	(e)	$\wp\wp\{a, b\}$
(c)	$\wp\emptyset$		

6. Given the sets $A, \ldots, G$ as in Exercise 1, list the members of each of the following:

(a)	$B \cup C$	(g)	$A \cap E$	(m)	$B - A$
(b)	$A \cup B$	(h)	$C \cap D$	(n)	$C - D$
(c)	$D \cup E$	(i)	$B \cap F$	(o)	$E - F$
(d)	$B \cup G$	(j)	$C \cap E$	(p)	$F - A$
(e)	$D \cup F$	(k)	$B \cap G$	(q)	$G - B$
(f)	$A \cap B$	(l)	$A - B$		

7. Given the sets in Exercise 1, assume that the universe of discourse is $\bigcup\{A, B, C, D, E, F, G\}$. List the members of the following sets:

(a) $(A \cap B) \cup C$	(h) $D' \cap E'$
(b) $A \cap (B \cup C)$	(i) $F \cap (A - B)$
(c) $(B \cup C) - (C \cup D)$	(j) $(A \cap B) \cup U$
(d) $A \cap (C - D)$	(k) $(C \cup D) \cap U$
(e) $(A \cap C) - (A \cap D)$	(l) $C \cap D'$
(f) G'	(m) $G \cup F'$
(g) $(D \cup E)'$	(n) $(B \cap C)'$

8. Let $A = \{a, b, c\}$, $B = \{c, d\}$ and $C = \{d, e, f\}$.

(a) What are:

(i) $A \cup B$	(v) $B \cup \emptyset$
(ii) $A \cap B$	(vi) $A \cap (B \cap C)$
(iii) $A \cup (B \cap C)$	(vii) $A - B$
(iv) $C \cup A$	

(b) Is a a member of $\{A, B\}$?

(c) Is a a member of $A \cup B$?

9. Show by using the set-theoretic equalities in Figure 1-7 for any sets A, B, and C,

(a) $((A \cup C) \cap (B \cup C')) \subseteq (A \cup B)$

(b) $A \cap (B - A) = \emptyset$

10. Show that the Distributive Law 4(a) is true by constructing Venn diagrams for $X \cup (Y \cap Z)$ and $(X \cup Y) \cap (X \cup Z)$.

11. The *symmetric difference* of two sets A and B, denoted $A + B$, is defined as the set whose members are in A or in B but not in both A and B, i.e.

$$A + B =_{def} (A \cup B) - (A \cap B)$$

(a) Draw the Venn diagram for the symmetric difference of two sets.

(b) Show that $A + B = (A - B) \cup (B - A)$ by means of the set-theoretic equalities in Figure 1-7. Verify that the Venn diagram for $(A - B) \cup (B - A)$ is equivalent to that in (a).

(c) Show that for all sets A and B, $A + B = B + A$.

The equations corresponding to (2-10) do not hold for relations (nor for functions which are not one-to-one correspondences). However, we have for any *one-to-one* relation $R: A \to B$:

$$(2\text{–}12) \quad \begin{aligned} R^{-1} \circ R &\subseteq id_A \\ R \circ R^{-1} &\subseteq id_B \end{aligned}$$

We should note here that our previous remarks about ternary, quaternary, etc. relations can also be carried over to functions. A function may have as its domain a set of ordered n-tuples for any n, but each such n-tuple will be mapped into a unique value in the range. For example, there is a function mapping each pair of natural numbers into their sum.

Exercises

1. Let $A = \{b, c\}$ and $B = \{2, 3\}$.
   (a) Specify the following sets by listing their members.
      (i) $A \times B$ (iv) $(A \cup B) \times B$
      (ii) $B \times A$ (v) $(A \cap B) \times B$
      (iii) $A \times A$ (vi) $(A - B) \times (B - A)$
   (b) Classify each statement as true or false.
      (i) $(A \times B) \cup (B \times A) = \emptyset$
      (ii) $(A \times A) \subseteq (A \times B)$
      (iii) $\langle c, c \rangle \subseteq (A \times A)$
      (iv) $\{\langle b, 3 \rangle, \langle 3, b \rangle\} \subseteq (A \times B) \cup (B \times A)$
      (v) $\emptyset \subseteq A \times A$
      (vi) $\{\langle b, 2 \rangle, \langle c, 3 \rangle\}$ is a relation from A to B
      (vii) $\{\langle b, b \rangle\}$ is a relation in A
   (c) Consider the following relation from A to $(A \cup B)$:
      $R = \{\langle b, b \rangle, \langle b, 2 \rangle, \langle c, 2 \rangle, \langle c, 3 \rangle\}$
      (i) Specify the domain and range of R
      (ii) Specify the complementary relation R' and the inverse R^{-1}
      (iii) Is $(R')^{-1}$ (the inverse of the complement) equal to $(R^{-1})'$ (the complement of the inverse)?

2. Let $A = \{a, b, c\}$ and $B = \{1, 2\}$. How many distinct relations are there from A to B? How many of these are functions from A to B? How many of the functions are onto? one-to-one? Do any of the functions have inverses that are functions? Answer the same questions for all relations from B to A.

3. Let

$$R_1 = \{\langle 1,1 \rangle, \langle 2,1 \rangle, \langle 3,4 \rangle, \langle 2,2 \rangle, \langle 3,3 \rangle, \langle 4,4 \rangle, \langle 4,1 \rangle\}$$

$$R_2 = \{\langle 3,4 \rangle, \langle 1,2 \rangle, \langle 1,4 \rangle, \langle 2,3 \rangle, \langle 2,4 \rangle, \langle 1,3 \rangle\}$$

(both relations in A, where $A = \{1, 2, 3, 4\}$).

   (a) Form the composites $R_2 \circ R_1$ and $R_1 \circ R_2$. Are they equal?

   (b) Show that $R_1^{-1} \circ R_1 \neq id_A$ and that $R_2^{-1} \circ R_2 \not\subseteq id_A$.

4. For the functions F and G in Figure 2-3:

   (a) show that $(G \circ F)^{-1} = F^{-1} \circ G^{-1}$.

   (b) Show that the corresponding equation holds for relations R and S in Figure 2-6.