CORRECTION TO LEMMA 2.2 OF THE FAST GAUSS TRANSFORM

VIKAS C. RAYKAR AND RAMANI DURAISWAMI

Abstract. In the paper on fast Gauss transform by Greengard and Strain [2] there is a mistake in Lemma 2.2 [page 83]. Here we present the corrected version of the Lemma. Similar reasoning can be applied to Lemma 2.3.

1. Incorrect Lemma 2.2. The Lemma 2.2 in [2](page 83) shows how to convert a Hermite expansion about s_B into a Taylor expansion about t_C.

LEMMA 1.1 (Lemma 2.2 in [2]). *The Hermite expansion*

$$G(t) = \sum_{\alpha \geq 0} A_\alpha h_\alpha \left(\frac{t - s_B}{\sqrt{\delta}} \right) \tag{1.1}$$

has the following Taylor expansion, about an arbitrary point t_C:

$$G(t) = \sum_{\beta \geq 0} B_\beta \left(\frac{t - t_C}{\sqrt{\delta}} \right)^\beta . \tag{1.2}$$

The coefficients B_β are given by

$$B_\beta = \frac{(-1)^{|\beta|}}{\beta!} \sum_{\alpha \geq 0} A_\alpha h_{\alpha+\beta} \left(\frac{s_B - t_C}{\sqrt{\delta}} \right). \tag{1.3}$$

If the A_α are given by (12), then the error $E_T(p)$ in truncating the Taylor series after p^d terms is bounded, in the box C with center t_C and side length $r\sqrt{2\delta}$, by

$$|E_T(p)| = \left| \sum_{\beta \geq p} B_\beta \left(\frac{t - t_C}{\sqrt{\delta}} \right)^\beta \right| \leq K Q_B \left(\frac{1}{p!} \right)^{d/2} \left(\frac{r^{p+1}}{1 - r} \right)^d , \tag{1.4}$$

if $r < 1$. The expression for B_β as given by Eq. 2.3 is incorrect. Also the error bound Eq. 2.4 is not valid. A new estimate for the error bound was given by [1]. We give the correct expression for B_β and the correct derivation for the error bound in the following lemma.

2. Correct Lemma 2.2. Following is the correct version of Lemma 2.2.

LEMMA 2.1 (Lemma 2.2 in [2]). *The Hermite expansion*

$$G(t) = \sum_{\alpha \geq 0} A_\alpha h_\alpha \left(\frac{t - s_B}{\sqrt{\delta}} \right) \tag{2.1}$$

has the following Taylor expansion, about an arbitrary point t_C:

$$G(t) = \sum_{\beta \geq 0} B_\beta \left(\frac{t - t_C}{\sqrt{\delta}} \right)^\beta . \tag{2.2}$$

The coefficients B_β are given by

$$B_\beta = \frac{1}{\beta!} \sum_{\alpha \geq 0} A_\alpha (-1)^{|\alpha|} h_{\alpha+\beta} \left(\frac{s_B - t_C}{\sqrt{\delta}} \right). \tag{2.3}$$

If the A_α are given by (12), then the error $E_T(p)$ in truncating the Taylor series after p^d terms is bounded, in the box C with center t_C and side length $r\sqrt{2\delta}$, by

$$|E_T(p)| = \left| \sum_{\beta \geq p} B_\beta \left(\frac{t - t_C}{\sqrt{\delta}} \right)^\beta \right| \leq \frac{Q_B}{(1-r)^d} \sum_{k=0}^{d-1} \binom{d}{k} (1 - r^p)^k \left(\frac{r^p}{\sqrt{p!}} \right)^{d-k}, \tag{2.4}$$

if $r < 1$.

Proof. The Hermite function $h_\alpha(t)$ has the following Taylor series expansion about an arbitrary point $t_0 \in \mathbf{R}^d$

$$h_\alpha(t) = \sum_{\beta \geq 0} \frac{(t - t_0)^\beta}{\beta!} (-1)^{|\beta|} h_{\alpha+\beta}(t_0). \tag{2.5}$$

Each Hermite function in Eq. 2.6 is expanded into a Taylor series about $(t_C - s_B)/\sqrt{\delta}$.

$$\begin{aligned} G(t) &= \sum_{\alpha \geq 0} A_\alpha h_\alpha \left(\frac{t - s_B}{\sqrt{\delta}} \right) \\ &= \sum_{\alpha \geq 0} A_\alpha \left[\sum_{\beta \geq 0} \frac{(-1)^{|\beta|}}{\beta!} \left(\frac{t - t_C}{\sqrt{\delta}} \right)^\beta h_{\alpha+\beta} \left(\frac{t_C - s_B}{\sqrt{\delta}} \right) \right] \\ &= \sum_{\beta \geq 0} \left[\frac{(-1)^{|\beta|}}{\beta!} \sum_{\alpha \geq 0} A_\alpha h_{\alpha+\beta} \left(\frac{t_C - s_B}{\sqrt{\delta}} \right) \right] \left(\frac{t - t_C}{\sqrt{\delta}} \right)^\beta \\ &= \sum_{\beta \geq 0} B_\beta \left(\frac{t - t_C}{\sqrt{\delta}} \right)^\beta, \end{aligned} \tag{2.6}$$

where,

$$B_\beta = \frac{(-1)^{|\beta|}}{\beta!} \sum_{\alpha \geq 0} A_\alpha h_{\alpha+\beta} \left(\frac{t_C - s_B}{\sqrt{\delta}} \right). \tag{2.7}$$

Using the symmetry condition $h_\alpha(t) = (-1)^{|\alpha|} h_\alpha(t)$, B_β can be written as,

$$B_\beta = \frac{1}{\beta!} \sum_{\alpha \geq 0} A_\alpha (-1)^{|\alpha|} h_{\alpha+\beta} \left(\frac{s_B - t_C}{\sqrt{\delta}} \right). \tag{2.8}$$

By the formula for A_α, we have

$$\begin{aligned} B_\beta &= \frac{1}{\beta!} \sum_{\alpha \geq 0} A_\alpha (-1)^{|\alpha|} h_{\alpha+\beta} \left(\frac{s_B - t_C}{\sqrt{\delta}} \right) \\ &= \frac{1}{\beta!} \sum_{\alpha \geq 0} \left[\frac{1}{\alpha!} \sum_{j=1}^{N_B} q_j \left(\frac{s_j - s_B}{\sqrt{\delta}} \right)^\alpha \right] (-1)^{|\alpha|} h_{\alpha+\beta} \left(\frac{s_B - t_C}{\sqrt{\delta}} \right) \\ &= \frac{1}{\beta!} \sum_{j=1}^{N_B} q_j \left[\sum_{\alpha \geq 0} \frac{(-1)^{|\alpha|}}{\alpha!} \left(\frac{s_j - s_B}{\sqrt{\delta}} \right)^\alpha h_{\alpha+\beta} \left(\frac{s_B - t_C}{\sqrt{\delta}} \right) \right] \\ &= \frac{1}{\beta!} \sum_{j=1}^{N_B} q_j h_\beta \left(\frac{s_j - t_C}{\sqrt{\delta}} \right). \end{aligned} \tag{2.9}$$

Using this expression 2.9 the truncation error can be bounded as,

$$
\begin{aligned}
|E_T(p)| &= \left| \sum_{\beta \geq p} B_\beta \left(\frac{t - t_C}{\sqrt{\delta}} \right)^\beta \right| \leq \sum_{\beta \geq p} |B_\beta| \left| \left(\frac{t - t_C}{\sqrt{\delta}} \right)^\beta \right| \\
&= \sum_{\beta \geq p} \frac{1}{\beta!} \left| \sum_{j=1}^{N_B} q_j h_\beta \left(\frac{s_j - t_C}{\sqrt{\delta}} \right) \right| \left| \left(\frac{t - t_C}{\sqrt{\delta}} \right)^\beta \right| \\
&\leq \sum_{\beta \geq p} \left[\sum_{j=1}^{N_B} |q_j| \left\{ \frac{1}{\beta!} \left| h_\beta \left(\frac{s_j - t_C}{\sqrt{\delta}} \right) \right| \right\} \right] r^\beta 2^{-\beta/2} \\
&= \sum_{\beta \geq p} \left[\sum_{j=1}^{N_B} |q_j| \left\{ \prod_{i=1}^{d} \frac{1}{\beta_i!} \left| h_{\beta_i} \left(\frac{(s_j)_i - (t_C)_i}{\sqrt{\delta}} \right) \right| \right\} \right] r^\beta 2^{-\beta/2} \\
&\leq \sum_{\beta \geq p} \left[\sum_{j=1}^{N_B} |q_j| \left\{ \prod_{i=1}^{d} \frac{1}{\sqrt{\beta_i!}} 2^{\beta_i/2} \right\} \right] r^\beta 2^{-\beta/2} \\
&= Q_B \sum_{\beta \geq p} \prod_{i=1}^{d} \frac{1}{\sqrt{\beta_i!}} r^{\beta_i} = Q_B \left[\sum_{\beta \geq 0} \prod_{i=1}^{d} \frac{1}{\sqrt{\beta_i!}} r^{\beta_i} - \sum_{\beta < p} \prod_{i=1}^{d} \frac{1}{\sqrt{\beta_i!}} r^{\beta_i} \right] \\
&= Q_B \left[\prod_{i=1}^{d} \left\{ \sum_{\beta_i \geq 0} \frac{1}{\sqrt{\beta_i!}} r^{\beta_i} \right\} - \prod_{i=1}^{d} \left\{ \sum_{\beta_i < p} \frac{1}{\sqrt{\beta_i!}} r^{\beta_i} \right\} \right] \\
&= Q_B \left[\left\{ \sum_{\beta_i < p} \frac{1}{\sqrt{\beta_i!}} r^{\beta_i} + \sum_{\beta_i \geq p} \frac{1}{\sqrt{\beta_i!}} r^{\beta_i} \right\}^d - \left\{ \sum_{\beta_i < p} \frac{1}{\sqrt{\beta_i!}} r^{\beta_i} \right\}^d \right] \\
&= Q_B \sum_{k=0}^{d-1} \binom{d}{k} \left(\sum_{\beta_i < p} \frac{1}{\sqrt{\beta_i!}} r^{\beta_i} \right)^k \left(\sum_{\beta_i \geq p} \frac{1}{\sqrt{\beta_i!}} r^{\beta_i} \right)^{d-k} \\
&\leq Q_B \sum_{k=0}^{d-1} \binom{d}{k} \left(\sum_{\beta_i < p} r^{\beta_i} \right)^k \left(\frac{r^p}{\sqrt{p!}} \sum_{\beta_i \geq 0} r^{\beta_i} \right)^{d-k} \\
&\leq Q_B \sum_{k=0}^{d-1} \binom{d}{k} \left(\frac{1 - r^p}{1 - r} \right)^k \left(\frac{r^p}{\sqrt{p!}} \frac{1}{1 - r} \right)^{d-k} \\
&= \frac{Q_B}{(1 - r)^d} \sum_{k=0}^{d-1} \binom{d}{k} (1 - r^p)^k \left(\frac{r^p}{\sqrt{p!}} \right)^{d-k} \qquad (2.10)
\end{aligned}
$$

□

REFERENCES

[1] B. J. C. Baxter and G. Roussos, *A new error estimate of the fast gauss transform*, SIAM J. Sci. Stat. Comput., 24 (2002), pp. 257–259.

[2] L. Greengard and J. Strain, *The fast gauss transform*, SIAM J. Sci. Stat. Comput., 12 (1991), pp. 79–94.