Computing Small Hitting Sets for Convex Ranges.

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Abstract

Let \( S \) be a set of \( n \) given points in \( \mathbb{R}^2 \). If \( A \) is a convex subset of \( \mathbb{R}^2 \) its “size” is defined as \(|A \cap S|\), the number of points of \( S \) it contains. We describe an \( O(n(\log n)^4) \) algorithm to find points \( z_1 \neq z_2 \), at least one of which must meet any convex set of size greater than \( 4n/7 \); \( z_1 \) and \( z_2 \) comprise a hitting set of size two for such convex ranges. This algorithm can then be used to construct (i) three points, one of which must meet any convex set of size \( > 8n/15 \); (ii) four points, one of which must meet any convex set of size \( > 16n/31 \); (iii) five points, one of which must meet any convex set of size \( > 20n/41 \).

Let \( S \) be a set of \( n \) given points in general position in \( \mathbb{R}^2 \). If \( A \) is a convex subset of \( \mathbb{R}^2 \), its “size” is defined to be \(|A \cap S|\), the number of points of \( S \) that it contains. The (Tukey) depth of a point \( z \in \mathbb{R}^2 \) is defined as the minimum (over all halfspaces \( h \) containing \( z \)) of \(|S \cap h|\), the size of the smallest halfspace containing \( z \). It is familiar that there always exists a point \( z \in \mathbb{R}^2 \) (not necessarily in \( S \)) with depth \( d(z) \geq n/3 \). Such a point is called a centerpoint for \( S \). The constant \( c = 1/3 \) is best-possible: for every \( c > 1/3 \) there are sets \( S \) with respect to which NO point has depth \( cn \). The interesting algorithm of Jadhav and Mukhopadhyay [2] computes a centerpoint in linear time.

Alternatively, if \( z \) is a centerpoint for \( S \), every convex set of size \( > 2n/3 \) MUST contain \( z \). A centerpoint may thus be said to “hit” all convex subsets of \( \mathbb{R}^2 \) with more than \( 2/3 \) of the points of \( S \). For this reason, centerpoint \( z \) is called a hitting-set (of size 1) for convex sets of size \( > 2n/3 \). Mustafa and Ray [4], following related work of Aronov et. al. [1], studied the possibilities for hitting sets with more than one point, a natural extension of the notion of centerpoint. They showed that given \( S \subseteq \mathbb{R}^2 \) there are points \( z_1 \neq z_2 \) (not necessarily in \( S \)) such that every convex set of size \( > 4n/7 \) must meet at least one of them. In addition they showed the constant \( 4/7 \) to be best possible for hitting sets of size 2: for every \( c < 4/7 \) there are sets \( S \) for which, whatever pair \( x \neq y \) be chosen, there is a convex subset containing \( > cn \) points of \( S \), but containing neither \( x \) nor \( y \). In [1] it had been shown that the optimal constant \( c \) was in the interval \([5/9, 5/8]\).

Let \( c_k \in (0, 1) \) be the smallest constant for which, given any set \( S \) of \( n \) points in \( \mathbb{R}^2 \), there are distinct points \( z_1, \ldots, z_k \), at least one of which must meet any convex set of size \( > c_kn \). We know \( c_1 = 2/3 \) and \( c_2 = 4/7 \). Mustafa and Ray were able to show that \( c_3 \in (5/11, 8/15) \), that \( c_4 \leq 16/31 \) and that \( c_5 \leq 20/41 \).

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Here we address some algorithmic questions about finding small hitting sets. The details are contained in the following statement, and its proof.

**Theorem 1** Let $S$ be a set of $n$ given points in general position in $\mathbb{R}^2$ and take $c_2 = 4/7$. Then in $O(n \log n)^4$, distinct points $z_1, z_2$ may be found so that if $A$ is a convex set of size $> c_2 n$, at least one of these points is in $A$.

Consider the set $\mathcal{R}$ of all convex subsets of size $> c_2 n$. For each pair $A \neq B$ in $\mathcal{R}$ consider $A \cap B$. Note that $|A \cap B| > n/7$, so there is a point $p_{A,B} = (u, v) \in A \cap B$ of minimal $y$–coordinate. The existence proof in [4] showed that $z_1$ may be taken as such a point, but one for a pair $A', B'$ where $p_{A',B'} = (u, v)$ has $v$ as large as possible (a point in the intersection of two ranges whose lowest point is highest). They also showed that $z_2$ may then be taken as the (usual) centerpoint for $S \setminus (A' \cap B')$ and everything works out.

Let $p = (u, v)$ be the lowest point in $A' \cap B'$ - the intersection of two ranges, each of size at least $c_2 n$ - where $v$ is as large as possible (its a highest lowest point). The proof of the theorem relies on understanding what such a point looks like in the line arrangement dual to $S$. We combine this with tools introduced by Matoušek [3] to compute $z_1$ in the stated complexity. Once we have $z_1$, $z_2$ - the centerpoint of $S \setminus (A' \cap B')$ - can be found in linear time.

It is easy now to show

**Corollary 1** As in Theorem 1, in $O(n \log n)^4$ we can find (i) distinct points $z_1, z_2, z_3$, one of which must meet any convex set of size $> 8n/15$; (ii) points $z'_1, z'_2, z'_3, z'_4$, one of which must meet any convex set of size $> 16n/31$; (iii) points $z''_1, z''_2, z''_3, z''_4, z''_5$, one of which must meet any convex set of size $> 20n/41$.

**References**


