# Sum of Squared Edges for MST of a Point Set in a Unit Square

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## 1 Introduction

Let the *weight* of a tree be the sum of the squares of its edge lengths. Given a set of points $P$ in the unit square let $W(P)$ be weight of the minimum spanning tree of $P$. If $P$ is simply the four corners of the square, then $W(P) = 3$. Gilbert and Pollack [2] demonstrated that $W(P) = O(1)$ and this was extended to an arbitrary number of dimensions by Bern and Eppstein [1]. While more recent divide-and-conquer approaches have shown that $W(P) \leq 4$, no point sets are known with $W(P) > 3$, and hence it has been widely conjectured (e.g. see [3]) that $W(P) \leq 3$. Here we show that $W(P) < 3.411$.

For a point set $P$ in a unit square, $MST(P)$ denotes a minimum spanning tree of $P$. Let $MST_k(P)$ denote the subgraph of $MST(P)$ in which all edges of length greater than $k$ have been removed from $MST(P)$. For any given point $X \in P$, define $MST_k(X, P)$ to be the connected component of $MST_k(P)$ containing $X$. Let $\boxplus$ be the corners of the unit square.

**Lemma 1.** $W(P) \leq W(P \cup \boxplus)$.

**Lemma 2.** *No edge in* $MST(P \cup \boxplus)$ *has length greater than* 1.

## 2 Result

By Lemma 1 it suffices to consider only point sets that include the corners of an enclosing unit square.

Kruskal's MST construction algorithm considers all edges defined by $P$ in sorted order. When an edge is considered, it is added to the existing graph only if it doesn't create a cycle. Let $e_m$ be the $m^{\text{th}}$ edge added. At step $m = 0$ no edges have been added and at step $m = |P| - 1 =: M$, $MST(P)$ is complete.

At each step of Kruskal's algorithm, each connected component of edges is a tree. We define $t_X^m$ to be the tree at step $m$ that contains point $X$. It helps to initialize the algorithm at $m = 0$ by letting every point $X$ of $P$ be a singular tree $t_X^0$ that is augmented when $X$ is an endpoint of an edge that is added. We also initialize $e_0 = 0$. Notice that $t_X^m = MST_{|e_m|}(X, P)$. Let $\mathcal{CH}(t)$ denote the vertices of the convex hull of a tree $t$. If $X$ is on $\mathcal{CH}(t_X^m)$, let $\angle^m(X)$ be the range of angles for which it is extreme. We set $\angle^0(X) = [0°, 360°]$. Over time this range of angles is reduced, and may have size 0 if $X$ is no longer on the hull. At any time $m$, for any given connected component $Z$, the set of all $\angle^m(X)$ for each point $X \in Z$ partitions the angle range [0, 360].

With this in place we define the region $C^m(X)$. If at time $m$, $X$ is on $\mathcal{CH}(t_X^m)$ and extreme in some range $[\alpha, \beta] = \angle^m(X)$ then $C^m(X)$ is the sector of a circle centered at $X$, with radius $\frac{|e_m|}{2}$ and spanning the angle range $[\alpha, \beta]$. If $X$ is not on the hull of $t_X^m$, then $C^m(X)$ is empty. Let $C^{*m}(X)$ be the union of all the sectors that $X$ has defined up to step $m$; that is $C^{*m}(X) = \cup_{\mu=0}^m C^\mu(X)$.

For a tree $t_X^m$, we define the region $A_X^m$ as follows.

$$A_X^m = \bigcup_{Y \in t_X^m} C^{*m}(Y).$$

$A_X^m$ is contained in the union of discs of radius $\frac{|e_m|}{2}$ centered on all points of $P$ in the same component as $X$. Points in different components have distance greater than $|e_m|$, otherwise an edge between them would have already been added. Thus if $A_X^m$ and $A_Y^m$ are different, then they are disjoint. Let $A^m$ denote union of all these regions and let $\Phi^m$ denote the area of all such regions defined at time $m$.

At time $m$, there are $|P| - m$ trees. Recall that points of $P$ not yet joined to other points are also considered to be trees. Let $\ell^m = |e_m|^2$.

**Lemma 3.** $\Phi^{m+1} = \Phi^m + \frac{\pi}{4}(|P| - m)(\ell^{m+1} - \ell^m)$.

*Proof.* At time $m$, each point $X$ on $\mathcal{CH}(t_X^m)$ has a sector $C^m(X)$ with radius $\frac{|e_m|}{2} = \frac{\sqrt{\ell^m}}{2}$. From our definition of $C^m(X)$, the sectors of all points on the

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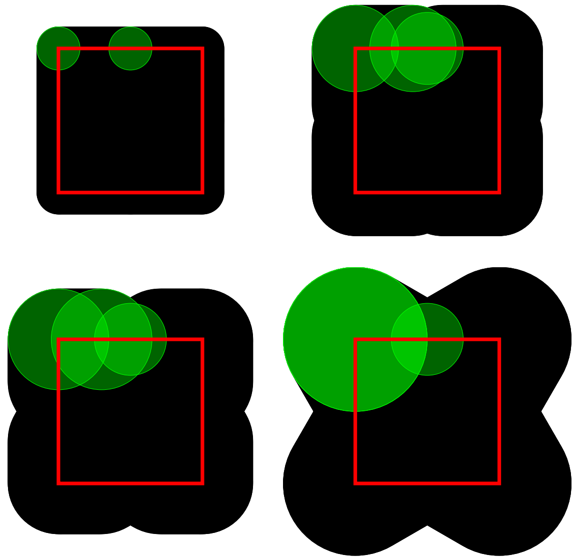

Figure 1: Depiction of $R$ for $d \in \{0.3, 0.6, 0.7, 1.0\}$.

convex hull of $t_X^m$ partition a circle of radius $\frac{\sqrt{\ell^m}}{2}$, which has area $\frac{\pi\ell^m}{4}$. From step $m$ to $m+1$, the radius of each of these sectors increases to $\frac{\sqrt{\ell^{m+1}}}{2}$ and the total area of the partitioned circle increases to $\frac{\pi\ell^{m+1}}{4}$. There are $|P|-m$ trees that each have this growth, and whose regions are disjoint, so multiplying the difference $\frac{\pi}{4}(\ell^{m+1}-\ell^m)$ from each tree by the number of trees, $|P|-m$, gives the result. □

Let $W^m$ denote the sum of the weights of all trees at time $m$.

**Lemma 4.** $W^m = \frac{4}{\pi}\Phi^m - (|P|-m)\ell^m$.

*Proof.* We induct on $m$. For the base case, $m = 0$, the spanning tree consists of no edges and all points are disconnected. Consequently, $W^0 = 0$, $\Phi^0 = 0$, and $\ell^0 = 0$. Assume that the statement holds for $W^m$. We will prove that it holds for $W^{m+1}$.

Because $e_{m+1}$ is the edge added at step $m+1$, we start with $W^{m+1} = W^m + \ell^{m+1}$. By the induction hypothesis, we substitute $W^m$ to obtain $\frac{4}{\pi}\Phi^m - (|P|-m)\ell^m + \ell^{m+1}$. By substitution, using Lemma 3, this equals $\frac{4}{\pi}\Phi^{m+1} - (|P|-m)\ell^{m+1} + \ell^{m+1}$. Simple rearranging yields the claimed result. □

**Lemma 5.** *Let $d$ denote $|e_M|$. If $d \leq \frac{1}{2}$, $\Phi^M \leq 2d + \frac{\pi d^2}{4} + 1$. Otherwise, $\Phi^M \leq d^2\sqrt{3} - \frac{1}{\sqrt{3}} + \frac{5\pi d^2}{12} + 4(d-d^2) + 1$.*

*Proof.* In Figure 1, we depict a region $R$ that we claim covers $A^M$. For every point $x$ on each edge $e$ of the square, define a circle of radius $\min\{\frac{d}{2}, \frac{f}{2}\}$, where $f$ is the distance from $x$ to the farther endpoint of $e$. This circle is meant to represent a specific radius of the growing circle that corresponds to $e_m$ as the algorithm progresses. If $x$ were a point in $P$, then its circle would intersect the equivalent growing circles centered on the endpoints of $e$. Therefore $x$ would no longer be on the convex hull of its component, after the two endpoints join. This would further imply that no sector of $x$ could keep expanding. We define $R$ to be the union of all such circles centered on the boundary of the square, together with the square region itself. This represents an upper bound on the region that $A^M$ can occupy, as the extreme case occurs when points in $P$ are located on the boundary of the square.

It remains to show that $R$ cannot grow any more on account of points of $P$ inside the square. Suppose that an interior point $y$ grows some sector $C^m(y)$ that contributes towards $\Phi^M$ outside $R$. Without loss of generality let this extra contribution be closest to the top edge $e$ of the square. Just like above, $C^M(y)$ can only grow above $e$ if $y$ is part of the upper hull of $t_y^m$ and that cannot happen if $y$ is in the same component as both endpoints of $e$. Let $x$ be the orthogonal projection of $y$ on $e$ and assume without loss of generality that the endpoint of $e$ farthest to $y$ is the right endpoint $r$. Therefore the endpoint of $e$ farthest to $x$ is also $r$. Furthermore, the midpoints of $\overline{xr}$ and $\overline{yr}$ have the same $x$-coordinate. Therefore, the portion of $C^M(y)$ above $e$ is contained in the circle of radius $\min\{\frac{d}{2}, \frac{f}{2}\}$ centered at $x$, which contradicts the assumption. All that remains is to calculate the area of $R$. This can be done algebraically but details are omitted from this version. □

**Theorem 6.** *For any set of points $P$ in the unit square, $W(P) \leq \frac{3\sqrt{3}+4}{\pi} - \frac{1}{\pi\sqrt{3}} + \frac{2}{3} \approx 3.4101$.*

*Proof.* From Lemma 1, we can assume that $P$ includes the corners of its enclosing unit square. $W^M = W(P)$, and by Lemma 4 is equal to $\frac{4}{\pi}\Phi^M - \ell^M$. This in turn is bounded in terms of $d$ in Lemma 5. Combining, we obtain the following upper bounds on $W(P)$ in terms of $d$: $\frac{4d^2\sqrt{3}+16(d-d^2)+4}{\pi} - \frac{4}{\pi\sqrt{3}} + \frac{5d^2}{3} - d^2$ when $d > 0.5$ and $\frac{8d-4}{\pi}$ for $d \leq 0.5$. This function is monotonically increasing for $0 \leq d \leq 1$, so substituting $d = 1$ and simplifying gives the claimed bound. □

## References


[1] M. W. Bern and D. Eppstein. Worst-case bounds for subadditive geometric graphs. In *Symposium on Computational Geometry*, pages 183–188, 1993.

[2] E. N. Gilbert and H. O. Pollack. Steiner minimal trees. *SIAM Journal in Applied Mathematics*, 16(1):1–29, 1968.

[3] D. B. West. `http://www.math.uiuc.edu/~west/regs/mstsq.html`.