Sum of Squared Edges for MST of a Point Set in a Unit Square

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1 Introduction

Let the weight of a tree be the sum of the squares of its edge lengths. Given a set of points \( P \) in the unit square let \( W(P) \) be weight of the minimum spanning tree of \( P \). If \( P \) is simply the four corners of the square, then \( W(P) = 3 \). Gilbert and Pollack [2] demonstrated that \( W(P) = O(1) \) and this was extended to an arbitrary number of dimensions by Bern and Eppstein [1]. While more recent divide-and-conquer approaches have shown that it has been widely conjectured (e.g. see [3]) that no point sets are known with weight greater than \( W \).

For a point set \( P \) in a unit square, \( MST(P) \) denotes a minimum spanning tree of \( P \). Let \( MST_k(P) \) denote the subgraph of \( MST(P) \) in which all edges of length greater than \( k \) have been removed from \( MST(P) \). For any given point \( X \in P \), define \( MST_k(X, P) \) to be the connected component of \( MST_k(P) \) containing \( X \). Let \( \mathbb{B} \) be the corners of the unit square.

Lemma 1. \( W(P) \leq W(P \cup \mathbb{B}) \).

Lemma 2. No edge in \( MST(P \cup \mathbb{B}) \) has length greater than 1.

2 Result

By Lemma 1 it suffices to consider only point sets that include the corners of an enclosing unit square.

Kruskal’s MST construction algorithm considers all edges defined by \( P \) in sorted order. When an edge is considered, it is added to the existing graph only if it doesn’t create a cycle. Let \( e_m \) be the \( m \)th edge added. At step \( m = 0 \) no edges have been added and at step \( m = |P| - 1 =: M \), \( MST(P) \) is complete.

At each step of Kruskal’s algorithm, each connected component of edges is a tree. We define \( t_X^m \) to be the tree at step \( m \) that contains point \( X \). It helps to initialize the algorithm at \( m = 0 \) by letting every point \( X \) of \( P \) be a singular tree \( t_X^0 \) that is augmented when \( X \) is an endpoint of an edge that is added. We also initialize \( e_0 = 0 \). Notice that \( t_X^0 = MST_{e_0}(P, X) \). Let \( CH(t) \) denote the vertices of the convex hull of a tree \( t \). If \( X \) is on \( CH(t_X^m) \), let \( \angle X \) be the range of angles for which it is extreme. We set \( \angle X = [0, 360] \). Over time this range of angles is reduced, and may have size 0 if \( X \) is no longer on the hull. At any time \( m \), for any given connected component \( Z \), the set of all \( \angle X \) for each point \( X \in Z \) partitions the angle range \([0, 360]\).

With this in place we define the region \( C^m(X) \). If at time \( m \), \( X \) is on \( CH(t_X^m) \) and extreme in some range \([\alpha, \beta]\) then \( C^m(X) \) is the sector of a circle centered at \( X \), with radius \( \frac{|e_m|}{2} \) and spanning the angle range \([\alpha, \beta]\). If \( X \) is not on the hull of \( t_X^m \), then \( C^m(X) \) is empty. Let \( C^m(X) \) be the union of all the sectors that \( X \) has defined up to step \( m \); that is \( C^m(X) = \cup_{\mu \leq m} C^\mu(X) \).

For a tree \( t_X^m \), we define the region \( A_X^m \) as follows.

\[
A_X^m = \bigcup_{Y \in t_X^m} C^m(Y).
\]

\( A_X^m \) is contained in the union of discs of radius \( \frac{|e_m|}{2} \) centered on all points of \( P \) in the same component as \( X \). Points in different components have distance greater than \( |e_m| \), otherwise an edge between them would have already been added. Thus if \( A_X^m \) and \( A_Y^m \) are different, then they are disjoint. Let \( A^m \) denote union of all these regions and let \( \Phi^m \) denote the area of all such regions defined at time \( m \).

At time \( m \), there are \(|P| - m\) trees. Recall that points of \( P \) not yet joined to other points are also considered to be trees. Let \( \ell^m = |e_m|^2 \).

Lemma 3. \( \Phi^{m+1} = \Phi^m + \frac{\pi}{4}(|P| - m)(\ell^{m+1} - \ell^m) \).

Proof. At time \( m \), each point \( X \) on \( CH(t_X^m) \) has a sector \( C^m(X) \) with radius \( \frac{|e_m|}{2} = \sqrt{\ell^m} \). From our definition of \( C^m(X) \), the sectors of all points on the...
convex hull of $t_m^m$ partition a circle of radius $\frac{\sqrt{m}}{2}$, which has area $\frac{\pi m}{2}$. From step $m$ to $m+1$, the radius of each of these sectors increases to $\frac{\sqrt{m+1}}{2}$ and the total area of the partitioned circle increases to $\frac{\pi (m+1)}{2}$. There are $|P| - m$ trees that each have this growth, and whose regions are disjoint, so multiplying the difference $\frac{\pi}{2}(t^{m+1} - t^m)$ from each tree by the number of trees, $|P| - m$, gives the result.  

Let $W_m$ denote the sum of the weights of all trees at time $m$.

**Lemma 4.** $W_m = \frac{4}{\pi} \Phi^m - (|P| - m) \ell^m$.

**Proof.** We induct on $m$. For the base case, $m = 0$, the spanning tree consists of no edges and all points are disconnected. Consequently, $W^0 = 0$, $\Phi^0 = 0$, and $\ell^0 = 0$. Assume that the statement holds for $W^m$. We will prove that it holds for $W^{m+1}$.

Because $c_{m+1}$ is the edge added at step $m+1$, we start with $W^{m+1} = W^m + \ell^{m+1}$. By the induction hypothesis, we substitute $W^m$ to obtain $\frac{4}{\pi} \Phi^m - (|P| - m) \ell^m + \ell^{m+1}$. By substitution, using Lemma 3, this equals $\frac{4}{\pi} \Phi^{m+1} - (|P| - m) \ell^{m+1} + \ell^{m+1}$. Simple rearranging yields the claimed result. 

**Lemma 5.** Let $d$ denote $|c_M|$. If $d \leq \frac{1}{2}$, $\Phi^M \leq 2d + \frac{5d^2}{3} + 1$. Otherwise, $\Phi^M \leq d^2 \frac{\sqrt{3}}{1} + \frac{5d^2}{3} + 4(d - d^2) + 1$.

**Proof.** In Figure 1, we depict a region $R$ that we claim covers $A^M$. For every point $x$ on each edge $e$ of the square, define a circle of radius $\min\{\frac{d}{2}, \frac{f}{2}\}$, where $f$ is the distance from $x$ to the farther endpoint of $e$. This circle is meant to represent a specific radius of the growing circle that corresponds to $c_m$ as the algorithm progresses. If $x$ were a point in $P$, then its circle would intersect the equivalent growing circles centered on the endpoints of $e$. Therefore $x$ would no longer be on the convex hull of its component, after the two endpoints join. This would further imply that no sector of $x$ could keep expanding. We define $R$ to be the union of all such circles centered on the boundary of the square, together with the square region itself. This represents an upper bound on the region that $A^M$ can occupy, as the extreme case occurs when points in $P$ are located on the boundary of the square.

It remains to show that $R$ cannot grow any more on account of points of $P$ inside the square. Suppose that an interior point $y$ on $e$ and assume without loss of generality that the endpoint of $e$ farthest to $y$ is the right endpoint $r$. Therefore the endpoint of $e$ farthest to $x$ is also $r$. Furthermore, the midpoints of $\overline{xe}$ and $\overline{ye}$ have the same $x$-coordinate. Therefore, the portion of $C^M(y)$ above $e$ is contained in the circle of radius $\min\{\frac{d}{2}, \frac{f}{2}\}$ centered at $x$, which contradicts the assumption. All that remains is to calculate the area of $R$. This can be done algebraically but details are omitted from this version.

**Theorem 6.** For any set of points $P$ in the unit square, $W(P) \leq \frac{3\sqrt{3} + 4}{\pi} \frac{1}{\sqrt{3}} + \frac{2}{3} \approx 3.4101$.

**Proof.** From Lemma 1, we can assume that $P$ includes the corners of its enclosing unit square, $W^M = W(P)$, and by Lemma 4 is equal to $\frac{4}{\pi} \Phi^M - \ell^M$. This in turn is bounded in terms of $d$ in Lemma 5. Combining, we obtain the following upper bounds on $W(P)$ in terms of $d$: $\frac{4d^2\sqrt{3} + 16(d - d^2)}{\pi} - \frac{4}{\pi\sqrt{3}} + \frac{5d^2}{3} - d^2$ when $d > 0.5$ and $\frac{8d - 4}{\pi}$ for $d \leq 0.5$. This function is monotonically increasing for $0 \leq d \leq 1$, so substituting $d = 1$ and simplifying gives the claimed bound.

**References**

