Packing disks that touch the boundary of a square

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Abstract. It is is shown that the total perimeter of n pairwise disjoint disks lying in the unit square $U = [0,1]^2$ and touching the boundary of U is $O(\log n)$, and this bound is the best possible.

1 Introduction

Given a collection of geometric objects $\mathcal{O}$, and a container $U \subseteq \mathbb{R}^d$, a *packing* is a finite set of translates of objects from $\mathcal{O}$ that are pairwise disjoint and lie in the container C. Extremal properties of packings (e.g., the densest packing of unit balls) are classical problems in discrete geometry. We consider a new variant of the problem related to TSP with neighborhoods (TSPN).

In the *Euclidean Traveling Salesman Problem* (ETSP), given a set S of n points in $\mathbb{R}^d$, we wish to find closed polygonal chain (*tour*) of minimum Euclidean length whose vertex set is S. The Euclidean TSP is known to be NP-hard, but it admits a PTAS in $\mathbb{R}^2$. In the *TSP with Neighborhoods* (TSPN), given a set of n sets (neighborhoods) in $\mathbb{R}^d$, we wish to find a closed polygonal chain of minimum Euclidean length that has a vertex in each neighborhood. The neighborhoods are typically simple geometric objects such as disks, polygons, line segments, or lines. TSPN is also NP-hard; it admits a PTAS for certain types of neighborhoods [5], but is hard to approximate for others [1].

For n connected (possibly overlapping) neighborhoods in the plane, TSPN can be approximated with ratio $O(\log n)$ by the algorithm of Mata and Mitchell [4]. At its core, the $O(\log n)$-approximation relies on the following early result by Levcopoulos and Lingas [3]: every (simple) rectilinear polygon P with n vertices, r of which are reflex, can be partitioned into rectangles of total perimeter $\text{per}(P) \log r$ in $O(n \log n)$ time.

One approach to approximate TSPN (in particular, it achieves a constant-ratio approximation for unit disks) is the following. Given a set S of n neighborhoods, compute a maximal subset $I \subseteq S$ of pairwise disjoint neighborhoods (i.e., an independent set), compute a good tour for I, and then augment it by traversing the boundary of each set in I. Since each neighborhood in $S \setminus I$ intersects some neighborhood in I, the augmented tour visits all objects in S. This approach is particularly appealing since good approximation algorithms are often available for pairwise disjoint neighborhoods [5]. The bottleneck of this approach is extending a tour of I by the total perimeter of the objects in I. This lead us to the following problem [2] (see Fig. 1):

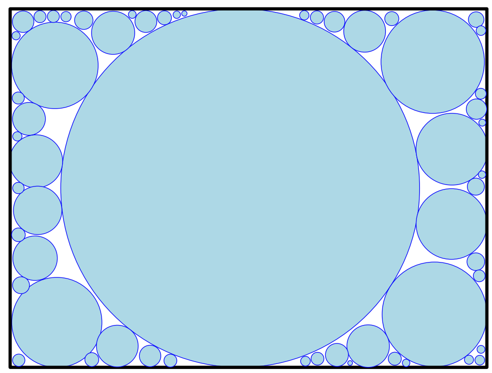

Figure 1: A packing of disks in a rectangle, with all disks touching the boundary.

Given a simple polygonal domain P in the plane and n disjoint disks lying in P and touching the boundary of P, what is the maximum ratio of the total perimeter of the disks and the perimeter of P? We address this problem in the simple setting where P is a unit square.

Theorem 1 *The total perimeter of n pairwise disjoint disks lying in the unit square $U = [0,1]^2$ and touching the boundary of U is $O(\log n)$. Apart from the constant factor, this bound is the best possible.*

2 Proof of Theorem 1

It is enough to bound the total diameter of n disks.

Upper bound. Let S be a set n disjoint disks in the unit square $U = [0,1]^2$ that touch the bottom side of U. Shrink each disk $D \in S$ by a factor $\rho \in (\frac{1}{2}, 1]$ from its common point with the x-axis such that its radius becomes $1/2^k$ for some integer $k \in \mathbb{N}$. The disks remain disjoint, they still touch the bottom side of U, and each radius decreases by a factor of at most 2. Partition the resulting disks into subsets as follows. For $i = 1, \ldots, \lfloor \log_2 n \rfloor$, let S_i denote the set of disks of radius $1/2^i$; and let S_0 be the set of disks of radius less than $1/n$. The sum of diameters of the disks in S_i,

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$i = 1, \ldots, \lfloor \log_2 n \rfloor$, is at most 1, since their horizontal diametrical segments are collinear and disjoint. The sum of diameters of the disks in S_0 is at most 2 since there are at most n disks altogether. Hence the sum of diameters of all original disks is at most $2(2 + \lfloor \log_2 n \rfloor)$, as required.

Lower bound construction. We construct a packing of $O(n)$ disks in the unit square $[-\frac{1}{2}, \frac{1}{2}] \times [0,1]$ such that every disk touches the x-axis, and the sum of their diameters is $\Omega(\log n)$. To each disk we associate its vertical *projection interval* (on the x-axis). The algorithm greedily chooses disks of monotonically decreasing radii such that (1) every diameter is $1/16^k$ for some $k \in \mathbb{N}_0$; and (2) if the projection intervals of two disks overlap, then one interval contains the other.

For $k = 0, 1, \ldots, \lfloor \log_{16} n \rfloor$, denote by S_k the set of disks of diameter $1/16^k$, constructed by our algorithm. We recursively allocate a set $X_k \subset [-\frac{1}{2}, \frac{1}{2}]$ to S_k, and then choose disks in S_k such that their projections intervals lie in X_k. Initially, $X_0 = [-\frac{1}{2}, \frac{1}{2}]$, and S_0 contains the disk of diameter 1 inscribed in $[-\frac{1}{2}, \frac{1}{2}] \times [0,1]$. The length of each maximal interval $I \subseteq X_k$ will be a multiple of $1/16^k$, so I can be covered by projection intervals of interior-disjoint disks of diameter $1/16^k$ touching the x-axis. Every interval $I \subseteq X_k$ will have the property that any disk of diameter $1/16^k$ whose projection interval is in I is disjoint from any (larger) disk in S_j, $j < k$.

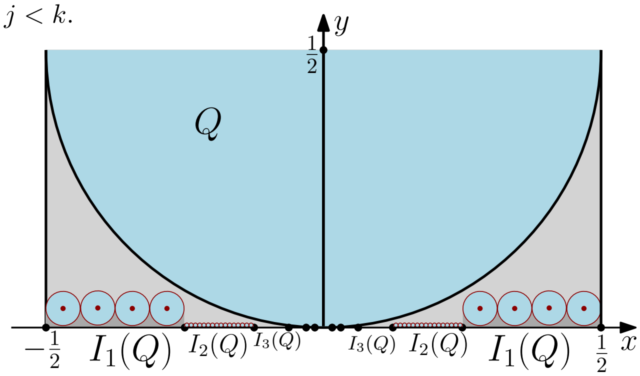

Figure 2: Disk Q and the exponentially decreasing pairs of intervals $I_k(Q)$, $k = 1, 2, \ldots$.

Consider the disk Q of diameter 1, centered at $(0, \frac{1}{2})$, and tangent to the x-axis (see Fig. 2). It can be easily verified that: (i) the locus of centers of disks tangent to both Q and the x-axis is the parabola $y = \frac{1}{2}x^2$; and (ii) any disk of diameter $1/16$ and tangent to the x-axis whose projection interval is in $I_1(Q) = [-\frac{1}{2}, -\frac{1}{4}] \cup [\frac{1}{4}, \frac{1}{2}]$ is disjoint from Q. Similarly, for all $k \in \mathbb{N}$, any disk of diameter $1/16^k$ and tangent to the x-axis whose projection interval is in $I_k(Q) = [-\frac{1}{2^k}, -\frac{1}{2^{k+1}}] \cup [\frac{1}{2^{k+1}}, \frac{1}{2^k}]$ is disjoint from Q. For an arbitrary disk D tangent to the x-axis, and an integer $k \geq 1$, denote by $I_k(D) \subseteq [-\frac{1}{2}, \frac{1}{2}]$ the pair of intervals corresponding to $I_k(Q)$; for $k = 0$, $I_k(D)$ consists of only one interval.

We can now recursively allocate intervals in X_k and choose disks in S_k ($k = 0, 1, \ldots, \lfloor \log_{16} n \rfloor$) as follows. Recall that $X_0 = [-\frac{1}{2}, \frac{1}{2}]$, and S_0 contains a single disk of unit diameter inscribed in the unit square $[-\frac{1}{2}, \frac{1}{2}] \times [0,1]$. Assume that we have already defined the intervals in X_{k-1}, and selected disks in S_{k-1}. Let X_k be the union of the interval pairs $I_{k-j}(D)$ for all $D \in S_j$ and $j = 0, 1, \ldots, k-1$. Place the maximum number of disks of diameter $1/16^k$ into S_k such that their projection intervals are contained in X_k. For a disk $D \in S_j$ ($j = 0, 1, \ldots, k-1$) of diameter $1/16^j$, the two intervals in X_{k-j} each have length $\frac{1}{2} \cdot \frac{1}{2^{k-j}} \cdot \frac{1}{16^j} = \frac{8^{k-j}}{2} \cdot \frac{1}{16^k}$, so they can each accommodate the projection intervals of $\frac{8^{k-j}}{2}$ disks in S_k.

We prove by induction on k that the length of X_k is $\frac{1}{2}$, and so the sum of the diameters of the disks in S_k is $\frac{1}{2}$, $k = 1, 2, \ldots, \lfloor \log_{16} n \rfloor$. The interval $X_0 = [-\frac{1}{2}, \frac{1}{2}]$ has length 1. The pair of intervals $X_1 = [-\frac{1}{2}, -\frac{1}{4}] \cup [\frac{1}{4}, \frac{1}{2}]$ has length $\frac{1}{2}$. For $k = 2, \ldots, \lfloor \log_{16} n \rfloor$, the set X_k consists of two types of (disjoint) intervals: (a) The pair of intervals $I_1(D)$ for every $D \in S_{k-1}$ covers half of the projection interval of D. Over all $D \in S_{k-1}$, they jointly cover half the length of X_{k-1}. (b) Each pair of intervals $I_{k-j}(D)$ for $D \in S_{k-j}$, $j = 0, \ldots, k-2$, has half the length of $I_{k-j-1}(D)$. So the sum of the lengths of these intervals is half the length of X_{k-1}; although they are disjoint from X_{k-1}. Altogether, the sum of lengths of all intervals in X_k is the same as the length of X_{k-1}. By induction, the length of X_{k-1} is $\frac{1}{2}$, hence the length of X_k is also $\frac{1}{2}$, as claimed. This immediately implies that the sum of diameters of the disks in $\bigcup_{k=0}^{\lfloor \log_{16} n \rfloor} S_k$ is $1 + \frac{1}{2} \lfloor \log_{16} n \rfloor$. Finally, one can verify that the total number of disks used is $O(n)$. Write $a = \lfloor \log_{16} n \rfloor$. Indeed, $|S_0| = 1$, and $|S_k| = |X_k|/16^{-k} = 16^k/2$, for $k = 1, \ldots, a$, where $|X_k|$ denotes the total length of the intervals in X_k. Consequently, $|S_0| + \sum_{k=1}^{a} |S_k| = O(16^k) = O(n)$, as required.

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