Burning the Medial Axes of 3D Shapes

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1 Introduction

The medial axis of an object, originally proposed by Blum [1], is the set of points having more than one closest point on the boundary of the object. It has been widely used as a shape descriptor due to its many properties, such as being "thin" (being one dimension lower than the object), homology equivalent to the object [3], and capturing the shape features.

The long term goal of our work is to develop the definitions of lower-dimensional shape descriptors as subsets of the medial axis (e.g., medial point of a 2D object, medial curve or point of a 3D object). These even "thinner" descriptors are useful in a range of applications such as shape alignment (using the medial point), 3D shape matching and deformation (using the medial curve).

There have been several definitions of a "medial point" on the medial axis of a 2D object, each defined as the local maximum of some function over the medial axis. The function proposed by Ogniewicz and Ilg [6], called *Potential Residue (PR)*, at a medial axis point x is the minimum length of the object boundary curve between the closest boundary points to x. The function proposed in our prior work [4], called *Extended Distance Function (EDT)*, equals the time at which x is burned away by a fire that is ignited from the ends of the medial axis curves and propagating geodesically along the curves at a constant speed. Both functions were shown to have a unique local maximum for a simply connected 2D object. The local maximum of EDT was also experimentally observed to be more stable than that of PR under boundary perturbations.

For a 3D object, the only mathematical definition of a medial curve that we are aware of was given by Dey and Sun [2]. The authors generalize the PR function to the medial axis of a 3D object (which they called the *Medial Geodesic Function (MGF)*), so that the value at a medial axis point x is the minimum geodesic distance on the object boundary between the closest boundary points to x. The medial curve is then made up of the singular set of MGF.

In this paper, we propose a new function definition over the medial axis of a 3D shape that generalizes the EDT over the medial axis of a 2D shape. The new function, which we call the *burn time (BT)*, captures the arrival time of a fire ignited from the border of the medial axis sheets and propagating geodesically along the sheets at constant speed. We prove several essential properties of BT that are analogous to EDT. As an on-going work, we are investigating the definition of medial curve based on the singular set of BT, which has the potential to be a more stable descriptor than the MGF-based definition [2].

2 The burn time function

Consider an object $S \subset \mathbb{R}^3$ whose medial axis M is a compact piecewise smooth cell complex. Let $f : \partial M \to \mathbb{R}$ be the Euclidean distance from points on ∂M to S. Note that f is a 1-Lipschitz function.

M in general is not a 2-manifold, but a collection of sheets joined at non-manifold curves and points. We decompose M into *manifold regions* (denoted by $M^{(2)}$), consisting of all points with a neighborhood in M which is homeomorphic to either an open disk (an interior point) or a half-open disk (on the boundary ∂M), *singular curves* (denoted $M^{(1)}$), consisting of all points with a neighborhood homeomorphic to a union of open and half-open disks identified along an arc, and *singular points* (denoted $M^{(0)}$). We refer to the union of $M^{(1)}$ and $M^{(0)}$ as the *singular set* of M, and use the notation $M^{(s)}$. We say a curve $\gamma : I \to M$ *does not cross the singular set* if it can be perturbed infinitesimally to a path which avoids the singular set entirely; otherwise, the curve *crosses the singular set.*

Burning on M proceeds similar to [4]. The fire is ignited from each point $x \in \partial M$ at time $f(x)$, and propagates geodesically on M at constant speed. The fire quenches as the fronts meet. When a fire front hits some point $x \in M^{(s)}$ such that x still has some un-burned disk neighborhood, the front dies out and

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does not propagate further. With the non-uniform ignition time, burning carries the distance to the object boundary. The dying-out rule ensures that the burning is not affected by small sheets in M arising from perturbations of the object boundary.

We give an explicit definition of burn time that is analogous to definition of the geodesic distance to the boundary. While the latter is the length of the shortest path, we consider the length of some shortest *path tree* that branches at the singular set $M^{(s)}$. Formally,

Definition 2.1 *A* path tree *for $x \in M$ is a map $t : T \to M$ where T is a rooted tree, t maps the root to x, and and every leaf of T is mapped to the boundary of M, such that:*

- *t maps any vertex of T to $M^{(s)}$.*
- *for every vertex v of T and disk D embedded in M with $t(v) \in D$, $t(T_v) \cap D \neq \{t(v)\}$ (where T_v is the subtree of T rooted at v)*
- *t maps every edge to a path that does not cross the singular set*

Intuitively, the branching rule means that each vertex of the path tree will have at least one outgoing child path that lies on each disk neighborhood of the vertex. We further define the *length* of a path tree as the supremum of that length (in M) of any path from the root to a leaf plus the function value f on that leaf. We then define *burn time (BT)* $BT_M(x) = \inf_{(t,T)} len(t,T)$. We say a path tree is *minimal* if it gives a path tree for every point in T, so for every $p \in T$, $BT_M(t(p)) = len(f, T_p)$.

BT is a generalization of the geodesic distance function from a manifold surface to a non-manifold surface. When M consists only of manifold regions, $BT_M(x)$ is the shortest geodesic distance from x to ∂M, and the minimal path tree at x is the shortest geodesic path (void of any branching vertices). When $M^{(s)}$ is non-empty, $BT_M(x)$ still behaves like a geodesic distance function in the manifold regions while exhibiting similar properties to EDT [4]:

Proposition 2.2 *$BT_M(x)$ has these properties:*

1. *It is 1-Lipschitz over a manifold region or along a singular curve.*
2. *It is upper semi-continuous everywhere. Furthermore, $BT_M(x) = \lim_{n\to\infty} BT_M(x_n)$ for some sequence $\{x_n\}$ converging to x.*
3. *It has no local minima away from ∂M.*
4. *$\{x \in M \mid BT_M(x) = \infty\}$ is equal to the maximal closed subcomplex of M.*

3 Future work

We are currently investigating an algorithm to compute BT over a discretization of the medial axis as a triangulated mesh. In this setting, a minimal path tree crosses a triangle or an edge for a bounded number of times, and hence we expect BT to be computable, for example using a front-advancing algorithm (akin to that for computing geodesic distances [5]). While it is natural to consider the singular set of BT as the medial curve, we have observed that this set alone may not preserve the homotopy of the medial axis, and we are investigating means to restore the homotopy by adding additional structures. Finally, it would be interesting to study the stability of both EDT and BT under boundary perturbations.

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