Medial Residue On a Piecewise Linear Surface

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1 Introduction

The medial axis of an object is a skeletal structure originally defined by Blum [1]. Formally, the medial axis of an object is the set of points having more than one closest point on the boundary of the object; alternatively, it can also be thought of as the set of centers of discs with maximal size that fit within the object, or in a variety of other ways. The medial axis is centered within the object, homology equivalent to the object if it is an open bounded subset of $\mathbb{R}^n$ [4], and (at least) one dimension lower than that of the object. These properties make the medial axis ideal for many applications including shape analysis and robotic path planning.

We are interested in defining a similar skeletal structure on a surface S that inherits the properties of the medial axis. Such a structure could then be used for applications such as shape analysis of surface patches as well as path planning in non-planar domains. We are particularly interested in piecewise smooth surfaces, which are more representative of typical outputs of discrete surface reconstruction algorithms (e.g., triangulated meshes) than globally smooth surfaces.

A natural approach would be to replace the Euclidean distances in the medial axis definition by geodesic distances over S. Indeed, Wolter [8] defines the *geodesic medial axis* on a smooth Riemannian manifold as the centers of *geodesic discs* with maximal size that fit in S. Interestingly, when S is only piecewise smooth, such an approach is not sufficient. Various definitions of the medial axis which are equivalent in $\mathbb{R}^n$ may not yield the same structure on S, and none of these structures guarantees the essential properties of the medial axis (being low-dimensional and homotopy preserving).

In this paper, we propose a new skeleton definition on a piecewise linear surface S, which we call the *medial residue*, and prove that the structure is a finite curve network that is homotopy equivalent to S. When S is a planar domain, the medial residue is equivalent to the medial axis, and so it is a natural extension of the medial axis onto surfaces. We also develop an efficient algorithm to compute the medial residue on a triangulated mesh, which builds on prior work to compute geodesic distances [5, 6].

2 The medial residue

Let S be a piecewise linear surface in $\mathbb{R}^3$. We first consider the set of points on S that do not have a unique direction for shortest geodesic paths to ∂S, denoted $\mathcal{M}^{SPD}$. Note that $\mathcal{M}^{SPD}$ reduces to the medial axis when S has no curvature. It is also not difficult to show that $\mathcal{M}^{SPD}$ is always a finite curve network. However, $\mathcal{M}^{SPD}$ may not preserve the homotopy of S around non-smooth, concave vertices (where the accumulative angle around the vertex is greater than 2π). For example, consider a concave vertex p that is in $\mathcal{M}^{SPD}$ and that has a neighborhood on S (which we call a *shadow*) such that the shortest path from any point x in the shadow to ∂S goes through p. Note that any point in the shadow would have a unique shortest path direction, and hence $\mathcal{M}^{SPD}$ would avoid the entire shadow, which can potentially cause disconnection in $\mathcal{M}^{SPD}$.

To achieve homotopy equivalence, we will add a curve subset of the shadows at concave vertices. While the actual geometry of these additional curve segments does not affect the topology of our medial residue, we would like these curves to be "centered", just like the medial axis. Naturally, we consider straight curves that bisect each shadow zone. We can formalize the notion of "straight" (which was proposed in [7]) and "bisect" as follows:

Definition 2.1. *We say a curve γ is* straight *if for every point $p \in \gamma$ the left and right curve angles at p are equal.*

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On a smooth surface, all geodesics are straight, and in fact this concept is equivalent to being a geodesic. However, on piecewise linear surfaces, there are geodesics that are not straight and straight curves that are not geodesic.

Definition 2.2. *A curve γ* bisects *a piecewise differentiable curve X at time t if $\gamma(t) \in X$ and the two angles bounded by γ and the tangent of X at $\gamma(t)$ are equal.*

Our medial residue is simply the union of $\mathcal{M}^{SPD}$ and the straight bisectors of the shadows, or formally:

Definition 2.3. *The* medial residue*, $\mathcal{MR}$ consists of any point $x \in S$ where we either have $x \in \mathcal{M}^{SPD}$ or where there are two distinct shortest paths from x to the boundary, γ_1 and γ_2, parameterized by arc length, which first intersect $\mathcal{M}^{SPD} \cup \partial S$ at $\gamma_1(t) = \gamma_2(t)$ such that $\gamma_1([0,t]) = \gamma_2([0,t])$ is straight and bisects $\mathcal{M}^{SPD} \cup \partial S$ at $\gamma_1(t)$.*

The usefulness of medial residue is reflected in the following theorem:

Theorem 2.4. *If S is a PL surface then $\mathcal{MR}$ is a finite curve network homotopy equivalent to S.*

3 Algorithm

Given a flat piecewise linear surface (i.e., a polyhedral surface), an algorithm exists that can compute, in $O(n^2 \log n)$ time for a mesh with n edges, a subdivision on each polyhedron face that captures the combinatorial structure of the distance function from a set of point sources [5, 6]. We first show that both the complexity and the correctness of the algorithm still hold when the point sources are replaced by edges on the boundary of the surface. Furthermore, we can show that the $\mathcal{M}^{SPD}$ consists of a subset of arcs and vertices in this subdivision, which can be identified in $O(n^2)$ time. Finally, the bisector curves (the second part of $\mathcal{MR}$) can be added in $O(n^2)$ time. Hence the end-to-end complexity of computing $\mathcal{MR}$ is $O(n^2 \log n)$.

4 The cut residue

The medial axis has strong connections to the cut locus [9]. However, even defining the cut locus on piecewise linear surfaces is a challenge, since the tangent space is not well defined on a non-smooth surface. The most commonly used definition of the cut locus of a point x in a non-smooth setting is the closure of the set of points that have two distinct geodesics to x. As with the medial axis, this definition gives problems when applied to a piecewise smooth manifold. As a result, algorithms for computing cut locus on a triangulated mesh either uses approximation [2] or are limited to convex surfaces [3]. By replacing ∂S in our medial residue definition with a point source x, we can similarly define a *cut residue* from x that is equivalent to the cut locus when S is a smooth surface. The homotopy and curve network properties in Theorem 2.4 can be shown to hold for cut residue when S is a piecewise linear surface, and the algorithm of medial residue can be easily adapted as well to allow efficient and exact computation of a cut-locus-like structure on arbitrary polyhedral surfaces.

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