Visibility Problems Concerning One-Sided Segments

Jonathan Lenchner* Eli Packer†

1 Introduction

We study a family of line segment visibility problems related to classical art gallery problems which are motivated by monitoring and surveillance requirements in commercial data centers. In traditional art gallery problems (see [4], [3] and [8]) an entire polygonal region must be kept under surveillance. In our case it is a prescribed collection of non-overlapping line segments in the interior of the polygon which must be kept under surveillance. Moreover, in many cases, it is required to see just one side of each segment. Some of our early results attacking simple variants of this problem were described in [2].

We consider distinct cases where the segments to be monitored are either all vertical, all axis-aligned, or alternatively, all arbitrarily aligned. Segments are assumed to be non-intersecting. Within these cases we identify several variants of the basic visibility problem. Namely, if visibility must be from a given side, but that side is specified by the problem poser, we say the problem is an instance of the **Poser's Choice** problem. If, on the other hand, the solver has the choice of which side to monitor the segment from, we say that it is an instance of the **Solver's Choice** problem. Variants of the Solver's Choice problem have been studied by Czyzowicz et al. [1], Toth [5]) and Urrutia [8]. A final variant is where the solver must monitor the entire segment from both sides (a variant also considered by Toth [5]).

In general we are interested in many aspects of these problems, from solving particular instances exactly, or with some approximation guarantee, to achieving hardness results, or achieving so-called combinatorial bounds, which say that for an arbitrary set of n segments, to see all segments using one of the visibility models may require some number, f(n), cameras. We consider both theoretical cameras with unlimited angular visibility and models of real cameras with some degree of restricted angular visibility or minimum/maximum depth of field restrictions. In [2] we showed that it was NP-hard to solve the Poser's Choice problem for the case of all vertical segments and cameras with limited angle of visibility. This result can be extended to cameras of unlimited angle of visibility in all the variants of the problem we have mentioned.

2 Results

In [2] we established hardness results for problems with either theoretical or realistic cameras. In this abstract we focus our discussion on describing some of what we know about the polynomial bounds for models involving theoretical cameras, i.e. cameras with unlimited angular visibility and no depth of field constraints. These results are summarized in Figure 1.

	Solver's Choice	Solver's Choice	Poser's Choice (All Segments from same specified side)	Poser's Choice	Both Sides
Vertical	U n/3 L n/3	U n/3 L n/3	U n/2 L n/2	U n/2 L n/2	U 2n/3 L 2n/3
Orthogonal	U n/2 (C) L n/3	U n/2 L n/3	U n/2 L n/2	U 3n/4 L 2n/3	U 4n/5 (T) L 2n/3
Arbitrary	U n/2 (T) L 2n/5 (U)	U 3n/4 L 2n/5 (U)	U 3n/4 L n/2	U 3n/4 L 2n/3	U 4n/5 (T) L 4n/5 (T)

Figure 1. A table summarizing what we know for the various problem variants. The 'U'-prefixed number in each cell denotes the best-known combinatorial upper bound, while the 'L'-prefixed number denotes the best-known lower bound. Results with a following (C) are due to Czyzowicz et al. [1], those with a following (T) are due to Toth [5] or [6], those with a following (U) are due to Urrutia [8], and the unlabeled ones are due to us.

In all cases we are looking for the minimum number of cameras that will suffice to see all segments in the worst case. All results are modulo additive constants. Thus the theoretical upper bounds are equal to the theoretical lower bounds. However, as the reader will undoubtedly notice, in most cases, there is a gap in our knowledge.

The results along the top row of Figure 1, for all vertical segments, were presented in [2], though at that time we did not make the more subtle distinctions between the two types of Solver's Choice and two types of Poser's Choice, so, in effect, only columns two, four and five were considered. As one moves from the top-left of this table to the bottom-right the problems become consistently harder. Thus the number of cameras required to solve cell (i, j) is less than or equal to the number of cameras needed to solve either cell $(i+1, j)$ or $(i, j+1)$ $\forall i, j$. Moreover, the same is true for any established upper and lower bounds. Our interest in the subtly different variations, e.g. of Solver's and Poser's Choice, is so that we can try to characterize precisely where the requirement for more cameras comes from as we move from the easier to harder problems. The lower bounds in the table are established by giving specific examples of segment configurations and arguing that (at least) the given number of cameras are required. The upper bounds are obtained by systematically proving that the given number of cameras can always be used to see the requisite number of segments. Additional problem gradations are possible.

Beginning in row 2 of Figure 1, Czyzowicz et al. [1] established the first interesting result: the upper bound of $\frac{n}{2}$ for the case of all axis-aligned segments under Solver's Choice where the Solver can

*IBM T.J. Watson Research Center
†IBM Haifa Research Lab

choose to see some points from one side and some from the other. The argument is an elegant exploitation of the following [7]:

Theorem. ***(Tutte)*** *A graph G has a perfect matching iff every subset of vertices S is such that the number of connected components of $G \setminus S$ of odd order is less than or equal to the number of vertices in S.*

The argument begins, WLOG, by extending the segments so that each end is within some common small epsilon of another edge, or the boundary. These segments give rise to a dissection of the original rectangle into "rooms" with tiny passageways between some adjacent pairs. Form a graph where the nodes of the graph are the rooms and there is an edge between nodes if there is a tiny passageway between them. Then use Tutte's Theorem to get a near-perfect matching of the rooms. Use the near perfect matching to situate a set of $\lceil \frac{n+1}{2} \rceil$ cameras at the passageway to each pair of rooms, which together see all points on each of the needed segments from one side or another.

A more careful, and consistent, camera placement enables one to extend the Czyzowicz et al. argument to give the identical bound for the more constrained Solver's Choice problem, as well as most-constrained Poser's Choice problem. For all the problems on orthogonal segments, gaps exist between the best known upper and lower bounds, except in the case of the most constrained Poser's Choice problem, where a lower bound of $\frac{n}{2}$ is carried over from the case of all vertical segments – just consider n vertical segments all spaced very close to one another and of height $h - \varepsilon$ (h being the height of the rectangle), where the poser requires you to see all segments from the left. Cameras can effectively see at most two segments entirely and a tiny bit of any other segment. Hence $\lceil \frac{n+1}{2} \rceil$ cameras are required.

The next interesting case we get to, and the only additional one we will consider in this short article, is that of Poser's Choice for the axis-aligned case. A simple example, establishing the $\frac{2n}{3}$ lower bound for this problem, pointed out to us four years ago by Toth [6], is given in Figure 2.

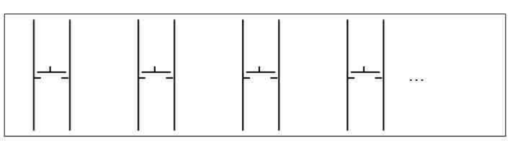

Figure 2. A set of segments for the full Poser's Choice problem requiring $\frac{2n}{3}$ cameras. The sides of the segments which need to be seen are indicated with little "ticky" marks - i.e. very tiny, orthogonally protruding line segments. The segments in each of the "H"s require two cameras for all of the specified segment sides to be seen entirely.

Finally, a $\frac{3n}{4}$ upper bound is established by virtue of the following:

Theorem. *Given n axis-aligned segments contained in a bounding rectangle, it is always possible to see the Poser's Choice of sides using at most $\lceil \frac{3n}{4} \rceil$ cameras.*

Proof. **(Sketch)** Extend the segments as in the Czyzowicz et al. argument, and again use the near-perfect matching to pair up rooms. Call matches of the form shown in Figure 3 "bad matches" since a single camera cannot entirely see all the needed segment sides. These are the only cases of two adjoining rooms in which a single camera does not suffice to see all required segment sides. Note that in each of these two examples, one can use *two* cameras to entirely see the needed segment sides in the two rooms, marked respectively A and B in each example. Moreover, it is easy to see that if the three called out lines, m, k, n are part of one bad match, where two cameras must be expended to see all needed segment sides, then they are not part of any additional bad matches. Thus there are at most $n/3$ bad matches in total.

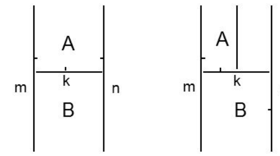

Figure 3. An "H" example (left) and an "h" example (right), in which a single camera cannot see all points on the required segment sides.

There are then two cases: (i) There are $b \leq \frac{n}{4}$ bad matches, or (ii) there are $\frac{n}{4} < b \leq \frac{n}{3}$ bad matches. In case (i) we use 2 cameras in each bad match and 1 camera in each good match. In case (ii) Suppose the are $\frac{n}{3} - h$ bad matches for $0 \leq h < \frac{n}{12}$. Use 2 cameras to see each of the 3 defining line segments (i.e. the analogs of m, n, k in Figure 3) and 1 camera to see each remaining line segment, In each case a computation shows that we use at most $\frac{3n}{4}$ cameras. □

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