Watchman Paths in Disk Grids

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Abstract

We study shortest watchman paths in rectangular arrangements of tangent unit-radius disks with disk centers on a square grid lattice. Upper and lower bounds are given for the length of shortest paths that see all of the boundary of each disk.

1 Introduction

The watchman tour problem in polygons involves finding a shortest tour so that every point in the polygon is seen from at least one point along the route [1, 2, 3]. We explore a related problem, which we will call the Disk Grid Path Problem: Given a rectangular arrangement of tangent unit-radius disks with disk centers on a square grid lattice, find the length of a shortest watchman path that sees all of the boundary of each disk.

An equivalent statement of the Disk Grid Path Problem is to find a shortest path with length $L(m,n)$ that travels through each of the $m \times n$ ($m \leq n$) "pockets" defined by the regions around the disks (see Figure 1).

Lemma 1.1. *A watchman path P with minimum length $L(m,n)$ is non-crossing.*

Proof. If P crosses itself, we can reroute the crossing segments locally so that the crossing is eliminated and the path remains connected. This local change yields a watchman path with strictly shorter length. □

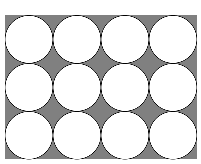

Figure 1: The 3×4 disk grid with 20 shaded pockets

We will assume that for the Disk Grid Path Problem, a given pocket is never visited twice. We leave the resolution of this conjecture as an open problem:

Conjecture 1.2 (The simplicity conjecture)**.** *An optimal solution to the* Disk Grid Path Problem *never visits the same pocket twice.*

Lemma 1.3. *Given the simplicity conjecture, the* Disk Grid Path Problem *is equivalent to finding a* Maximum-Turn Hamiltonian Path *in an $m \times n$ square grid graph.*

Proof. By the simplicity conjecture, an optimal path visits each vertex of the grid graph once. If the pocket is visited on a turn, the vertex contributes a cost of $\pi/2$ to the path, which is less than its cost of 2 if the pocket was visited on a straight path. There are always two terminating endpoints, so their contributions are equal and have cost 1. Thus, the shortest path is exactly the one with the most turns. □

Let $T(m,n)$ be the number of turns in a Maximum-Turn Hamiltonian Path of an $m \times n$ square grid graph. Then, the length of the shortest watchman path, $L(m,n)$, is given by

$$\begin{aligned} L(m,n) &= \frac{\pi}{2}T(m,n) + 2 + 2(nm - T(m,n) - 2) \\ &= 2(nm-1) - (2 - \frac{\pi}{2})T(m,n) \end{aligned}$$

Therefore, in order to find the minimum-length watchman path in disk grids, we bound the number of right-angle turns in a Hamiltonian path on an $m \times n$ grid graph.

2 Results

We summarize our results in Table 1, of lower and upper bounds on $T(m,n)$, $m \leq n$. This, in turn, bounds $L(m,n)$, via the above equation.

Conjecture 2.1. *$T(m,n) = nm - m$ for m,n even.*

2.1 Upper Bounds

Examine two adjacent vertices in the grid graph. There are three types of ways the vertices can be

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m	n	lower bound	upper bound
odd	$= m$	$nm - n - 1$	$nm - n - 1$
odd	odd\even	$nm - n$	$nm - n$
even	odd	$nm - m$	$nm - m$
even	even	$nm - m$	$nm - 4$

Table 1: Combinatorial bounds for $T(m, n)$

visited. The equivalence classes up to symmetry are in Figure 2 (square vertices are endpoints):

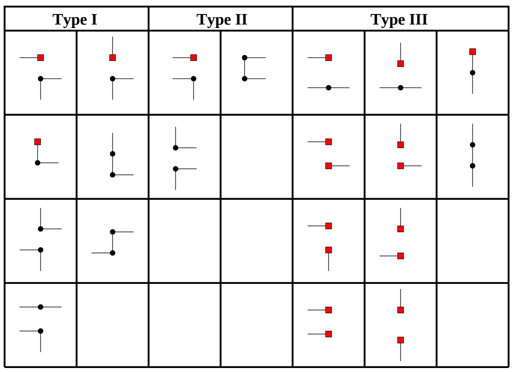

Figure 2: The types of vertex visitations

We divide the grid into horizontal strips of thickness 2, then analyze which vertex types can occur.

Lemma 2.2. *A horizontal strip of the grid graph that contains a right-pointing Type I pair of vertices must contain at least* 1 *additional non-turn vertex.*

Proof. In all cases, either a non-turn vertex appears, or a new Type I pair arises to the right side of the old pair. A Type I pair cannot appear on the right boundary, so there must be a non-turn vertex. □

Corollary 2.3. *If a strip contains a Type I pair then it contains at least two non-turn vertices.*

Proof. All Type I pairs either contain a non-turn vertex already, or are symmetric so that there is a non-turn both to the right and to the left of the pair. □

Lemma 2.4. *A strip with odd width must contain a Type I or Type III pair of vertices. Specifically, such a strip must contain at least* 2 *non-turn vertices.*

Proof. Each of the Type II vertices involves exactly two pairs of vertices so one pair must be left out. □

Theorem 2.5. *The upper bounds for $T(m, n)$ in Table 1 hold.*

Proof. Due to space constraints, we only show the case where m is odd and n is even. The remaining cases are similar. Take the $m \times n$ grid graph and cut it into $\frac{n}{2}$, $2 \times m$ strips. Since m is odd, by Lemma 2.4, there is a pair of non-turn vertices in each of the strips, leaving at most $2m - 2$ turns per strip. Thus, there are at most $\frac{n}{2}(2m - 2) = nm - n$ turns in an optimal path. □

2.2 Lower Bounds

Watchman paths that achieve the number of turns in Table 1 can be constructed (in general) via a spiraling pattern. Figure 3 gives several examples.

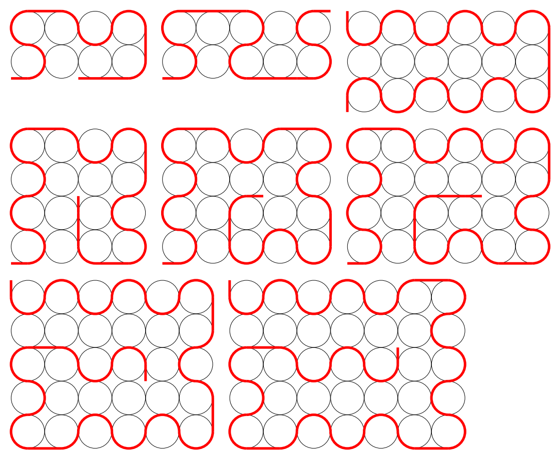

Figure 3: Examples of watchman paths that achieve the lower bounds given in Table 1

3 Conclusion

For future work, we would like to establish tight bounds for the unresolved case when both m and n are even. In addition, we would like to prove the simplicity conjecture for paths and consider the closely related Disk Grid Cycle Problem.

References

[1] W. Chin and S. Ntafos. Optimum watchman routes. *Information Processing Letters* 28(1):39-44, 1988.

[2] A. Dumitrescu and C. D. Tóth. Watchman tours for polygons with holes. *Computational Geometry: Theory and Applications* 45(7):326-333, 2012.

[3] S. Carlsson, H. Jonsson, and B. J. Nilsson. Finding the shortest watchman route in a simple polygon. *Discrete and Computational Geometry* 22(3):377-402, 1999.