Guarding simple polygons with semi-open edge guards *

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1 Introduction

Let $bd(P)$ denote the boundary of a simple polygon P. Points p and q of P are mutually visible if segment $\overline{pq}$ lies entirely inside P. This notion of visibility gave birth to an extensive literature on art-gallery problems [3], concerned with guarding the floor of a polygonal art-gallery. Chvatal [2] showed that $\lfloor n/3 \rfloor$ point guards are always sufficient and sometimes necessary. Allowing guards to move on an edge gives rise to the class of edge guard problems. An edge guard is closed (open) if the end-points of the edge are included (excluded), semi-open if only one end-point is included. Shermer [5] established an upper bound of $\lfloor 3n/10 \rfloor + 1$ on the number of closed edge guards needed, while Toussaint [3] showed that $\lfloor n/4 \rfloor$ guards are sometimes necessary. A *guard edge* is one that guards all of P. In [6], Toth et al. have shown that a non star-shaped simple polygon has at most one open guard edge. Park et al. [4] showed that such a polygon can have at most 3 closed guard edges.

Thus it is interesting to explore the scenario in which the edge guards are semi-open. A semi-open edge guard includes exactly one of the end-points. For clarity and focus, in this paper the included end-point is always the end that is met first in a clockwise traversal of P. We show that a non star-shaped polygon has at most 3 semi-open guard edges and propose an $O(n)$ algorithm to find all semi-open guard edges of a polygon.

2 Semi-open guard edges

Lemma 1 *Let $e = (u, v]$ be a semi-open edge of a polygon P and p a point interior to it. Then p is visible from e iff the set of common vertices of the paths $p \rightsquigarrow u$ and $p \rightsquigarrow v$ is either $\{p\}$ or $\{p, v\}$.*

Figure 1 shows that a non star-shaped polygon can have 3 semi-open guard edges. In fact, the following result holds.

Theorem 1 *Every non star-shaped simple polygon has at most three semi-open guard edges.*

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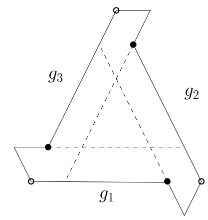

Figure 1: *A non-starshaped polygon that has three semi-open guard edges : g_1, g_2 and g_3*

3 Characterizing semi-open guard edges

Let r be a reflex vertex of P. With respect to a counter-clockwise order of $bd(P)$, let r^- be the vertex that precedes r on $bd(P)$, and r^+ the one that succeeds it. Let p^- be the intersection with $bd(P)$ of a ray shot from r in the direction $\overrightarrow{r^- r}$, while p^+ is the intersection with a ray shot from r in the direction $\overrightarrow{r^+ r}$.

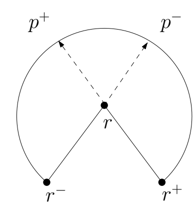

Figure 2: *Two subpolygons defined by a reflex vertex r*

These rays define two polygons: a left polygon $C_{left}(r)$ bounded by the chord rp^- and the part of $bd(P)$ from p^- to r in the counter-clockwise order; and a right polygon $C_{right}(r)$ bounded by the chord rp^+ and the part of $bd(P)$ from r to p^+ in the counter-clockwise order.

The left (respectively, right) kernel, $K_{left}(P)$ (respectively, $K_{right}(P)$) is the intersection of all the left (respectively, right) polygons $C_{left}(r)$ (respectively, $C_{right}(r)$), while the kernel of P is the intersection of $K_{left}(P)$ and $K_{right}(P)$.

Fact 1 *The kernels $K_{left}(P)$ and $K_{right}(P)$ are both convex.*

Toth et al. [6] showed that the left and right kernels can be used to define left-kernel and right-kernel decompositions of $int(P)$. These decompositions were used to prove the following theorem.

Theorem 2 *A open edge $e = (a, b)$ of a simple polygon P is a guard edge iff e intersects both the left and right kernels of P.*

Theorem 3 *A semi-open edge $e = (a, b]$ is a guard edge iff e has a non-empty intersection with $C_{left}(r) \cap C_{right}(r)$ for every reflex vertex, r.*

4 Algorithm

In [1], Bhattacharya et al. proposed a linear time algorithm for computing a shortest internal line segment l from which a polygon P is weakly internally visible. Central to their algorithm is the notion of a non-redundant component. Both $C_{left}(r)$ and $C_{right}(r)$, as defined in this paper, are components of P. A component is non-redundant if it does not properly contain any other component. They show how to compute all non-redundant components in linear time.

Let the ends of each non-redundant component that lie on $bd(P)$ be marked blue and red in counter-clockwise order in an initial counter-clockwise traversal of $bd(P)$.

To calculate the number of non-redundant components that an edge intersects, we obtain this value for the previous edge, subtract the number of red marks it contains, and then add the number of blue marks the current edge contains. To initialize the process, we find the number of non-redundant components for the first edge. This requires an extra pass over $bd(P)$ with a counter initialized to 0; every red mark passed over decrements the counter, and every blue mark increments it. This finds all closed guard edges. To narrow down to just semi-open guard edges, we remove all edges with both end points reflex.

At the end of the second round, we declare those edges e as semi-open guard edges whose intersection count *edge-Count(e)* is equal to the number of non-redundant components.

5 Polygons with holes

A polygon P with holes can have guard edges that lie on the outer boundary or on the boundaries of the holes. Park et al. [4] have shown that to establish upper bounds it is enough to consider polygons with only one convex hole, indeed just one triangular hole. It is quite obvious that no semi-open edge of this triangular hole can be an guard edge as it cannot see all the points on its own boundary. As for guard edges on the outer boundary, the following theorem of [4] for closed guard edges carries over when the guard edges are semi-open.

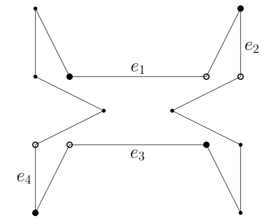

Figure 3: *Semi open edges $e_1 - e_4$ guard this polygon*

Theorem 4 *For a polygon P with a convex hole H, the number of guard edges is at most 3.*

6 Conclusion

By considering semi-open guard edges, we are led to some interesting conclusions. The upper bound on the number of semi-open guard edges is the same as for closed guard edges. A more careful characterization is needed for a semi-open guard edge as one or both the kernels can be empty. It would also be interesting to find tight upper and lower bounds on the number of semi-open edge guards needed to guard a polygon P. The classes of polygons in Figures 3, 4 seem to suggest a lower bound of $2n/7$ semi-open edge guards.

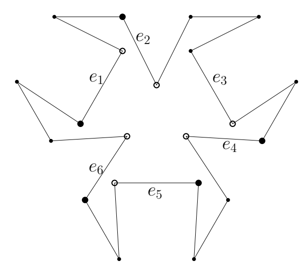

Figure 4: *Semi open edges $e_1 - e_6$ guard this polygon*

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