Point Set Isolation Using Unit Disks is NP-complete

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Abstract. We consider the situation where one is given a set S of points in the plane and a collection $\mathcal{D}$ of unit disks embedded in the plane. We show that finding a minimum cardinality subset of $\mathcal{D}$ such that any path between any two points in S is intersected by at least one disk is NP-complete. This settles an open problem raised in [1]. Using a similar reduction, we show that the Multiterminal Cut Problem introduced in [4] remains NP-complete when restricted to unit disk graphs.

1. Introduction and Main Result

In this note we show that the Point Set Isolation Problem defined below in Problem 1 is NP-complete. This problem was introduced in [1] where a polynomial-time constant-factor approximation algorithm was presented, but the problem complexity was stated as an open problem. As a motivation for studying this problem, its relevance to trap coverage in sensor networks is mentioned, where one wants to detect certain spacial transitions among the observed objects (see for example [3]).

In order to show NP-completeness of the Point Set Isolation Problem we are going to reduce the Planar Subdivision Problem defined in Problem 2 to it. This problem is NP-complete by Proposition 3.

Problem 1 (Point Set Isolation Problem [1])**.** *Given a set S of k points in the plane and a collection $\mathcal{D}$ of n unit disks embedded in the plane, no disk containing a point of S. The goal is to find a minimum cardinality subset $\mathcal{D}' \subseteq \mathcal{D}$, s.t. every path between two points in S is intersected by at least one disk in $\mathcal{D}'$.*

Problem 2 (Planar Subdivision Problem)**.** *Given a simple unweighted planar graph $G = (V, E)$ embedded in the plane and a set S of k points properly contained in the faces of G with no face containing more than one point, find the minimum cardinality set $E' \subseteq E$ such that in the embedding of the reduced graph $G' = (V, E')$, no two points are contained in the same face.*

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Proposition 3. *The Planar Subdivision Problem is* NP*-complete if k is not fixed.*

Theorem 4. *The Point Set Isolation Problem is* NP*-complete if k is not fixed.*

Problem 5 (Multiterminal Cut Problem [4])**.** *Given a simple graph $G = (V, E)$ and a set $S \subseteq V$ of k terminals, the task is to find the minimum cardinality set $E' \subseteq E$ such that in $G' = (V, E \setminus E')$ there is no path between any two nodes in S.*

Theorem 6. *The Multiterminal Cut Problem remains* NP*-complete on unit disk graphs if k is not fixed.*

For a high level description of the proof of Theorem 4, we reduce an instance $I_2 = (G_2, S_2)$ of the Planar Subdivision Problem in polynomial time to an instance $I_1 = (\mathcal{D}, S_1)$ of the Point Set Isolation Problem. We do this by first transforming the embedding of G_2 to an "equivalent" straight line embedding on an integer grid. Each embedded edge then gets replaced by an edge gadget which consists of a path of unit disks constructed in such a way that every edge gadget contains the same amount of unit disks, regardless of the length of the embedded edge. The dimensions of each edge gadget is chosen such that no two unit disks of different edge gadgets intersect. Furthermore, we replace each embedded vertex v by a vertex gadget which consists of a cycle of unit disks which is circularly arranged around v. Each edge gadget of edges incident to v will intersect a small number of disks contained in the vertex gadget. The main task of the reduction is to choose the radius of the disks and the dimension of the gadgets such that every edge gadget consists of the same amount of disks and so that non-incident edge gadgets are disjoint. For the proof of Theorem 6 similar gadgets are used.

References

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