

Packing identical simple polygons is NP-hard

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Abstract

Given a small polygon S , a big simple polygon B and a positive integer k , it is shown to be NP-hard to determine whether k copies of the small polygon (allowing translation and rotation) can be placed in the big polygon without overlap. Previous NP-hardness results were only known in the case where the big polygon is allowed to be non-simple. A novel reduction from PLANAR-CIRCUIT-SAT is presented where a small polygon is constructed to encode the entire circuit.

Introduction.

Packing is a fundamental problem in computational geometry. In this paper we study the problem of packing multiple copies of a small object inside a big object:

Simple Polygon Packing. *Given a small simple polygon S , a big simple polygon B , and a positive integer k , is it possible to place k copies (allowing translation and rotation) of the small polygon inside the big polygon without overlap?*

This problem was heretofore neither known to be in P, nor in NP, nor to be NP-hard. Here we show it is NP-hard.

Our result is the first to establish the hardness of packing of multiple copies of a simple polygon inside another simple polygon. Previous reductions for related problems fall into two categories. In the case of multiple small polygons, a reduction from KNAPSACK or PARTITION is easy. In the case of having a nonsimple big polygon, the reduction in [3] is from PLANAR-CIRCUIT-SAT. Such a reduction creates a big polygon which is essentially a drawing of the circuit, where the interior of the big polygon represents the wires and the gates, but where there are holes between all of the wires. Without the ability to literally create a big polygon that uses holes to create a circuit drawing, nothing was known. Our construction is also a reduction from PLANAR-CIRCUIT-SAT, but in a completely different manner. Previously the circuit was encoded in the big polygon; our big polygon is

independent of all aspects of the circuit, other than the circuit size, while the circuit is encoded entirely in the small polygon.

However, because our construction creates a small polygon which is nonconvex and polynomial in size, there remains a range of open problems relating to the packing of identical polygons. The simplest such variation (most likely to be in P) would be: given as the big polygon an orthogonally convex simple polygon drawn on a unit grid, how many grid-aligned 2×2 unit squares can be packed? (This is a slightly easier variant of problem 56 on the *Open Problems Project* [1], which was shown to be in NP in [2]).

Reduction overview. We give here a very high level description of our reduction. We reduce from a variant of circuit-sat where the circuit is drawn in a planar embedding on an $n \times n$ grid. The idea is to have our small polygons be unit-square-like where n^2 of them can only pack into a slightly larger than $n \times n$ -sized square-like large polygon if the packing is done according to a grid. The polygons deviate from being perfect squares by having several inclusions and exclusions; these force a small polygon to have certain interactions with the neighboring small polygon if a packing of n^2 small polygons is to be achieved. Each polygon will have a number of allowable vertical shifts, which among other things, represent a series of truth values. The construction is very carefully constructed such that the position of a small polygon relative to its neighbor will encode its position in the grid packing, its truth value, as well as the truth value of two neighboring small polygons. Given that this relative positioning encodes all this information, inclusions can be made to allow or deny shifts encoding configurations that are consistent with the given planar circuit on a grid.

- [1] Erik D. Demaine, Joseph S. B. Mitchell, and Joseph O'Rourke. The open problems project. <http://cs.smith.edu/~orourke/TOPP/Welcome.html>.
- [2] Dania El-Khechen, Muriel Dulieu, John Iacono, and Nikolaj van Omme. Packing 22 unit squares into grid polygons is NP-complete". In *CCCG*, pages 33–36, 2009.
- [3] Robert J. Fowler, Mike Paterson, and Steven L. Tanimoto. Optimal packing and covering in the plane are NP-complete. *Inf. Process. Lett.*, 12(3):133–137, 1981.



Figure 1: An illustration of our reduction. The 9 colored small polygons are all identical, and the inner white-black boundary defines the large polygon. Note that for ease of viewing the whitespace and other visual elements have been exaggerated dramatically, and certain elements of the construction have been simplified. The inclusions and exclusions on the large polygon form an arithmetic progression. To the lower left is a closeup of what we call the *nailer*; this forces a unique positioning depending on a small polygon's location in the grid. To the bottom-right is a closeup of a horizontal protrusion and inclusion; these are designed so that the notch occupied is a unique function of the position. By removing some notches, some configurations can be allowed or forbidden. The vertical groups of three are meant to illustrate states (such as true and false). For reasons described in the full paper, 64 such vertical groups are needed instead of the three illustrated.

