Tactix on an S-shaped Board

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Abstract

TACtIX is a geometric variant of the classic game NIM. A full characterization of a class of instances of TACtIX is presented where it is shown how to efficiently compute which player has a winning strategy.

Nim. The classic game NIM is traditionally played with some distinct piles of objects such as coins. There are two players which take turns alternately. Each player may take any number of objects from any pile (from one object to the entire pile), but may only take from the one pile that the player chooses for that turn. After each turn, the piles grow smaller, and eventually, all the objects are gone. There are two versions of NIM that differ on what happens then: NIM with the *normal play convention* has the player that takes the last counter win, while *misère* NIM has this player lose. NIM is a decidely one-dimnesional game; in this paper we consider a natural two-dimensional geometric variant of NIM.

TacTix. TACTIX, a game invented by Piet Hein, is a two-dimensional version of NIM. TACTIX is also a two-player game. There is a 4×4 grid of counters, and each player is allowed to take any horizontal or vertical sequence of consecutive counters. TACTIX is played with the misère rule, meaning that the player that takes the last counter loses. TACTIX has been solved; the second player has a winning strategy [1].

If the normal play convention were used, then this would be more obvious; the second player could make the 180° rotation of whatever move that the first player chose, and repeat until the last counter. This strategy would also work on a square grid of even size, or on a rectangular grid with both dimensions even. On a rectangular grid with one odd dimension, the first player has a winning strategy: take the entire middle column (or row, if there is an odd number of rows), which leaves two rectangular boards of equal size, and copy whichever move the second player chooses to make on one board on the other board. However, these strategies do not work in TACTIX, where the misère convention is used.

Tactix. TACtIX is a variation of TACTIX (note lower case t), played with the normal play convention (the rules regarding legal moves are the same). Since requiring a rectangular starting position makes this game trivially solved, any subset of a rectangular grid is an allowable starting position in Tactix.

Impartial Games. All three games (NIM, TACTIX, and TACtIX) share the property of the allowable moves depending only on the game position and not on which of the two players is on move. Such games are called *impartial games.* This condition negates the requirement of stating who is on move; each position has a specific result with reference to whoever is on move. Most games are not impartial; for instance, GO is not impartial, since a player can only play a stone of the color assigned to the player.

Finite Games. All three games also share the property of being finite. A finite game is one where, if it is played starting in any given position, a final result is always reached after no more than a number of moves that only depends on the starting position. In any of these three games, each move removes at least one counter (or object in the case of NIM), and the game is decided when there are no more counters (or objects), so the number of moves played is at most the number of counters. One way for a game not to be finite is if a position can be repeated by a sequence of moves without a rule governing what happens then. CHESS without the threefold repetition or 50 move rules is an example of a game that is not finite. However, with the 50-move or threefold repetition rules, the game of CHESS is finite.

Sprague-Grundy. TACTIX is equivalent to NIM, as are all impartial games with the normal play convention, by the Sprague-Grundy theorem [3, 4]. This gives each starting position a so-called NIM value, also known as the *nimber* or *Grundy value.* The NIM value of a position is given as the lowest nonnegative integer that is not the NIM value of any resulting position after any move (and it is zero for a position with no legal moves). There is an efficient way to calculate the NIM value of a disjoint combination of two or more of these games (where the games are played simultaneously and a legal move is selecting one game and making a legal move there), namely that the NIM value of the combination is the bitwise

XOR of the Nim values of the individual games. A Nim value gives a determination of whether or not the first player has a winning strategy: the first player has a winning strategy unless the Nim value is zero. This gives a recursive way to compute the Nim value of a position, but it can easily be exponential to compute for general positions in certain games, even with dynamic programming.

Thus the main question one can ask in the study of a particular game is, how efficiently can you determine which player has a winning strategy? For an impartial and finite game with the normal play convention, this can be done by deriving an algorithm to compute the Nim value of the game.

Monotonic boards. For Tactix boards of the form, which are shaped like a staircase and which we call *monotonic*:

×	···	×						
		×	···	×				
				···	···	···		
						×	···	×

where × represents a counter, there is a polynomial-time dynamic programming algorithm to compute the Nim value, similar to the solution to Linear Cram described in [2]. Removing any counters leaves two disjoint groups of counters, henceforth each move must be entirely in one group or the other. The Nim value of the resulting combination is the bitwise XOR of the Nim values of the two groups left. In the direct solution generated by the definition of the Nim value, there are only $O(n^2)$ (where n is the number of counters) possible connected groups, one per start and end point, the calculation of the Nim value of each only requires a polynomial number of lookups of Nim-values of smaller connected groups (as there are only $O(n^2)$ legal moves in any position). When such a Nim value is obtained, it is memoized, or stored in a table to be looked up later. This ensures that the Nim value of any given connected group is calculated only once. This solution thus runs in polynomial time, $O(n^4)$ if $\log n$ is word-sized.

If the vertical connections were entirely disallowed, then there is an easy solution, namely calculating the bitwise XOR of the number of counters on each row, since this is nothing more than Nim. This method does not work for monotonic Tactix, because of the possibility of a vertical move (taking two vertically-adjacent counters). This raises a question: is there a method of computing the Nim value that is faster than the dynamic programming algorithm? Our main result is that if the number of lines is limited to two, then the answer is yes. This is the type of starting position we call a *S-shaped board*:

×	···	×		
		×	···	×

Result. The Nim value of a Tactix game on an S-shaped board with a counters on the top and b counters on the bottom is:

$$r(a,b) = \begin{cases} r(a',b'), & r(a',b') < a'+b' \\ a'+b'+1, & r(a',b') = a'+b' \\ & \text{ and } a'+b' < m-1 \\ a+b, & r(a',b') = a'+b' \\ & \text{ and } a'+b' \geq m-1 \end{cases}$$

where m is the lowest power of 2 below or at a, $a' = a-m$, and $b' = b-m$. This only applies if the lowest power of 2 below or equal to b is also m; if not, then the Nim value is simply the number of counters.

This formula can be evaluated in time $O(\log n)$, where $n = a+b$.

The proof of this result can be found as Lemma 23 in the draft which was submitted with this paper, the proof of which occupies pages 23-42. Efficiently determining which side has a winning strategy for more general versions of Tactix remains open.

References

[1] M. Beeler. Taxtix. *HACKMEM*, 74, 1972.

[2] Erik D. Demaine. Playing games with algorithms: Algorithmic combinatorial game theory. *CoRR*, cs.CC/0106019, 2001.

[3] P. M. Grundy. Mathematics and games. *Eureka*, 1:6–8, 1939.

[4] R. P. Sprague. Uber mathematische kampfspiele. *Tohoku Mathematical Journal*, 41:438–444, 1935-36.