Transformations to critical arcgons: progress on tighter bounds for Delaunay stretch

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Abstract

Chew observed that the stretch factor of L_2 Delaunay triangulations was at least $\pi/2$ in the paper that also established the L_1 Delaunay as the first geometric spanner; recent examples have raised the lower bound to 1.5932. The upper bound of 2.45 established in 1992 has also recently come down to below 2 by Xia. We outline a program that brings both upper and lower bounds closer to 1.6 by characterizing the properties of examples that are maximal under local transformations. Here we describe the characterization and our plan of bounding the stretch factor of L_2 Delaunay triangulations.

1 Introduction

The *Delaunay stretch* of a point set S is the maximum, for all pairs $p, q \in S$, of the ratio of the length shortest path from p to q in the Delaunay triangulation of S over $|pq|$. It is known that the Delaunay stretch for any point set is upper bounded by a constant [3, 4]. The best bounds published [5, 6] until now are [1.5932, 1.998]. Our project aims to narrow this interval, by transforming examples into canonical form without decreasing stretch. Our plan is sketched in three section: In Sec. 2 we transform any given point set to an *arcgon*. In Sec. 3 we transform any arcgon to a *max arcgon*. In Sec. 4 we bound the stretch factors of max arcgons, thus bounding the Delaunay stretch.

2 Counterexamples and Arcgons

Fig. 1(a,b) shows example point sets with high Delaunay stretch. The points lie densely along the boundary of a union of discs. By controlled perturbation into general position, Delaunay edges in the disc interiors can be directed to preserve Delaunay stretch. This construction as a sequence of discs helps find lower bounds, but also helps compute upper bounds.

An *arcgon* is defined as an embedded graph whose

- edges are either circular arcs or straight lines.

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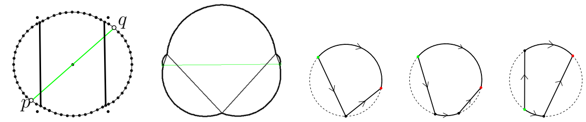

Figure 1: Point sets with high Delaunay stretch (a) $\kappa = 1.5846$ from [1], (b) our $\kappa = 1.59324$. Balanced (equal length) paths around three face types: (c) wedge, (d) anti-parallel stump, (e) parallel stump

- faces f are convex subsets of a disc c with vertices on the boundary of c in a sequence. The ends are circle segments, with special vertices p and q. Interior faces are *wedge* with 3 edges (Fig. 1(c)) or *stump* with 4 (Fig. 1(d,e)).
- outer boundary contains only circular arc edges. The edges in the interior, called *diagonals*, are straight line edges that connect the points of intersections of two neighboring discs.

Arcgons must satisfy two empty circle properties:

Local Delaunay: Vertices of face f lie on or outside of circles of neighboring faces of f.

pq-Delaunay: p, q on or outside circles of all faces.

In *realizable arcgons*, all diagonals intersect $\overline{pq}$. From a given point set S we can extract a realizable arcgon without decreasing Delaunay stretch, κ.

Lemma 1 *Let $p, q \in S$ attain the maximum Delaunay stretch κ in the Delaunay triangulation of the points S. There is a realizable arcgon for which $\overline{pq}$ has a stretch factor $\geq \kappa$.*

Proof Sketch: From the Delaunay triangulation of S, take the sequence of triangles that intersect the interior of segment $\overline{pq}$. Then in every triangle of this sequence, replace the edges that do not intersect the $\overline{pq}$ with the corresponding arcs of the circumcircle. Similar transformations have been used in [2, 4, 5]. ∎

3 Max Arcgons

Define the *complexity* of an arcgon to be the number of its faces. A *max arcgon* has stretch factor greater than all realizable arcgons of lower complexity, and not less than all realizable arcgons of equal complexity. Thus, Delaunay stretch is bounded by the stretch factor of max arcgons.

We now characterize max arcgons by studying local transformations

Lemma 2 *In a max arcgon A, the segment $\overline{pq}$ does not pass through the endpoints of any of the diagonals*

We say that an edge e of an arcgon is *critical* if some shortest path between p and q passes through e. We can show that in max arcgons all edges are critical. Here we sketch the proof for one case.

Lemma 3 *In a max arcgon A, the circular arc edges of every stump face f are critical*

Proof Sketch: Let L be the part of the arcgon to the left of the face f and R be the part to the right. If an arc-edge of f, say e, is not critical then we can increase the stretch factor κ of A by rotating R about the center, o, of the circle associated with the face f. We parameterize this rotation on the angle θ subtended by the arc-edge e at o. Let $(')$ be represent the derivative with respect to θ. We show that $\kappa'' > 0$ whenever $\kappa' = 0$ *i.e.* κ can be increased by increasing or decreasing θ. ∎

Using similar first derivative and second derivative analysis we can prove the following lemma:

Lemma 4 *Every max arcgon is realizable and*

- *The arc-edge of any wedge face is critical*
- *Diagonals adjacent to any stump face are critical*
- *Circular arc edges of two neighboring wedge faces cannot coincide.*

We are currently working to complete the proof that any diagonal adjacent to two wedge faces is critical. Together with Lem. 4, this would imply:

Claim 5 (To be established) *All the edges of a max arcgon are critical*

This claim constrains max arcgons to a class we call *critical arcgons*.

4 Upper bound on critical arcgons

In a *critical arcgon* interior faces are of three types:

balanced wedge: the length of the arc edge equals the sum of the two diagonals (Fig. 1(c)).

balanced anti-parallel stump: the difference in lengths of the two circular arc edges equals the sum of the two diagonals (Fig. 1(d)).

balanced parallel stump: the difference in lengths of the circular arc edges equals the difference in the diagonals (Fig. 1(e)).

Claim 6 *Let A be a critical arcgon, $\ell(A)$ be the length of the shortest path between the special vertices p, q of A. Let $d(A)$ be the length of the shortest geodesic path between p and q that passes through the faces of A. Let $L(A)$ be half of the perimeter of the arcgon A minus the end faces. Then,*

$$g(A) = \ell(A) - \frac{\pi}{2}d(A) - 0.04L(A) \leq 0$$

Note that if the critical arcgon is also realizable (as is the case with max arcgons) then $d(A)$ equals $|pq|$. Moreover $L(A) \leq \ell(A)$. Thus the above lemma immediately implies,

Theorem 7 *For a realizable critical arcgon A, the stretch factor*

$$\ell(A)/|pq| \leq \frac{\pi}{2(1-0.04)} \leq 1.636245$$

A construction of a critical arcgon raises the lower bound as well, so the maximum stretch factor of Delaunay triangulations would lie in [1.59324,1.63625].

Proof Sketch for Claim 6: The proof is by induction on the number of faces of the critical arcgon. The proof divides into three cases depending on whether the penultimate face (q is on the last face) is a balanced wedge, parallel, or anti-parallel stump. We can show that $g(A)$ attains a maximum when p lies on this penultimate face. The final inequality is established numerically. ∎

References

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