1 Homework 5 (Solution)

1. Given

$$\frac{1}{y-x_i} = \frac{1}{(y-x_*)-(x_i-x_*)} = \frac{1}{y-x_*}\left(1-\frac{x_i-x_*}{y-x_*}\right)^{-1}$$

Following the hint we can write

$$\frac{1}{y-x_i} = \frac{1}{y-x_*}\left(1+\frac{x_i-x_*}{y-x_*}+\ldots+\frac{(x_i-x_*)^{p-1}}{(y-x_*)^{p-1}}\right)+\frac{1}{y-x_*}\left(\frac{\left(\frac{x_i-x_*}{y-x_*}\right)^p}{1-\frac{x_i-x_*}{y-x_*}}\right)$$

The required S expansion is then

$$\Phi(x_i,y) = \frac{1}{y-x_i} = \sum_{m=0}^{p-1} b_m(x_i,x_*)\, S_m(y-x_*) + Error(p)$$

with

$$b_m(x_i,x_*) = (x_i-x_*)^m \quad \text{and} \quad S_m(y-x_*) = (y-x_*)^{-m-1},$$

and the residual term is given by

$$Error(p) = \frac{1}{y-x_*}\left(\frac{\left(\frac{x_i-x_*}{y-x_*}\right)^p}{1-\frac{x_i-x_*}{y-x_*}}\right) = \frac{(x_i-x_*)^p}{(y-x_*)^p}\frac{1}{y-x_i}$$

We are given $x_*=0$. So the terms can be written as

$$b_m(x_i,x_*) = x_i^m, \quad S_m(y-x_*) = y^{-m}, \quad Error(p) = \frac{x_i^p}{y^p}\frac{1}{y-x_i}$$

Our goal is now to evaluate a relationship between the maximum magnitude of the residual, and p and l. We are given $x_i \in \left[-\frac{1}{2}l, \frac{1}{2}l\right]$, $y \in \left[\frac{3}{2}l, \frac{5}{2}l\right]$. Looking at the error funtion we see that, given that y is positive increasing y should decrease the function monotonically. Because we wish to be conservative in our error estimates we can set $y=\frac{3}{2}l$ and obtain

$$Error(p) \le \frac{x_i^p}{\left(\frac{3}{2}\right)^p l^p}\frac{1}{\frac{3}{2}l-x_i}.$$

The situation with x is a bit more complex as x has both positive and negative values. We can find the minimum of the function of p with x_i

$$\frac{\partial E}{\partial x_i} = \frac{p x_i^{p-1}}{y^p}\frac{1}{y-x_i}+\frac{x_i^p}{y^p}\left(-\frac{1}{(y-x_i)^2}\right)(-1) = \frac{p x_i^{p-1}}{y^p}\frac{1}{y-x_i}+\frac{x_i^p}{y^p}\left(\frac{1}{y-x_i}\right)^2$$
$$= \frac{x_i^{p-1}}{y^p(y-x_i)}\left(p+\frac{x_i}{y-x_i}\right) = \frac{x_i^{p-1}}{\left(\frac{3}{2}\right)^p l^p\left(\frac{3}{2}l-x_i\right)}\left(p+\frac{x_i}{\frac{3}{2}l-x_i}\right)$$

This vanishes when

$$x_i = -p\left(\frac{3}{2}l-x_i\right)$$

or

$$(p-1)\,x_i = \frac{3p}{2}l, \quad \text{or} \quad x_i = \frac{3p}{2(p-1)}l$$

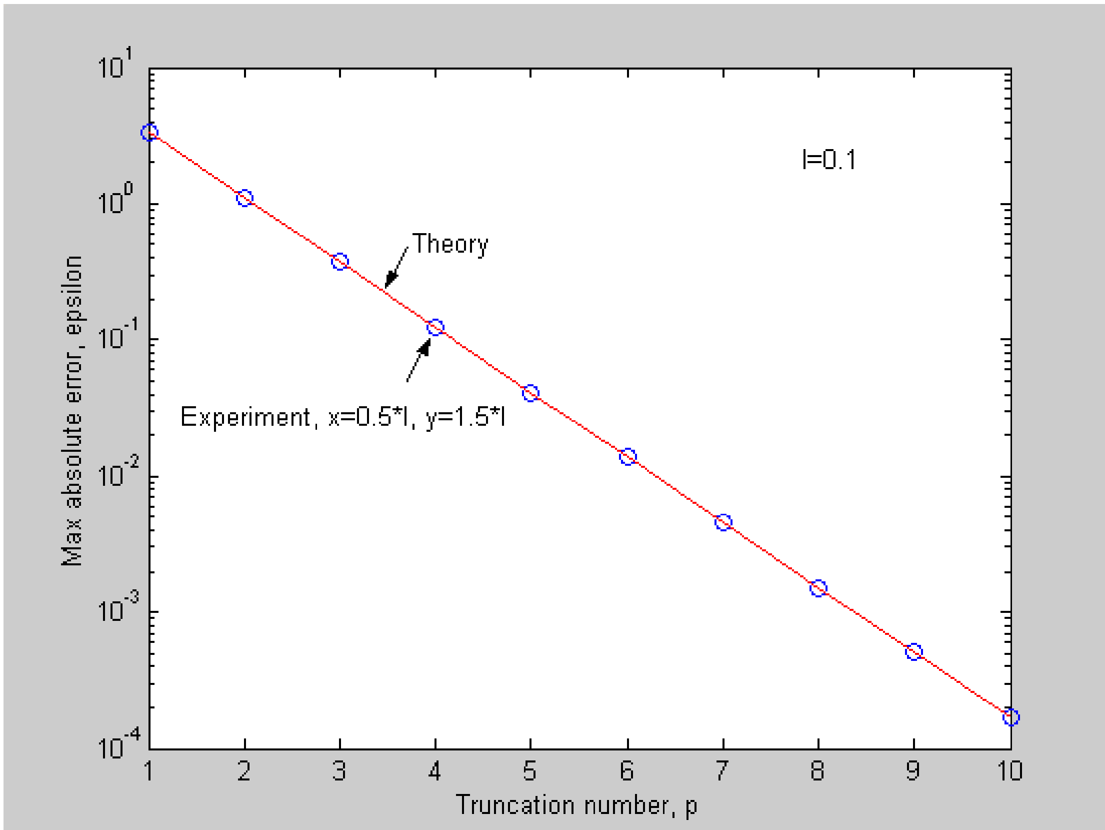


Figure 1:

For large p this is close to $1.5l$ while at $p = 2$ it is at $3l$, and so the extremum is attained outside the range $\left[-\frac{1}{2}l, \frac{1}{2}l\right]$. We can therefore bound the error by choosing the largest value of the numerator and the smallest one of the denominator

$$E_S(p) \leq \frac{\left(\frac{1}{2}\right)^p l^p}{\left(\frac{3}{2}\right)^p l^p} \frac{1}{\frac{3}{2}l - \frac{1}{2}l} = \frac{1}{3^p l} = \epsilon.$$

So we have

$$\log \epsilon = -p \log 3 - \log l,$$

or

$$p = \left[-\frac{\log \epsilon + \log l}{\log 3}\right] = \left[\log_3 \frac{1}{\epsilon l}\right],$$

and p can be chosen as the nearest integer above this estimate.

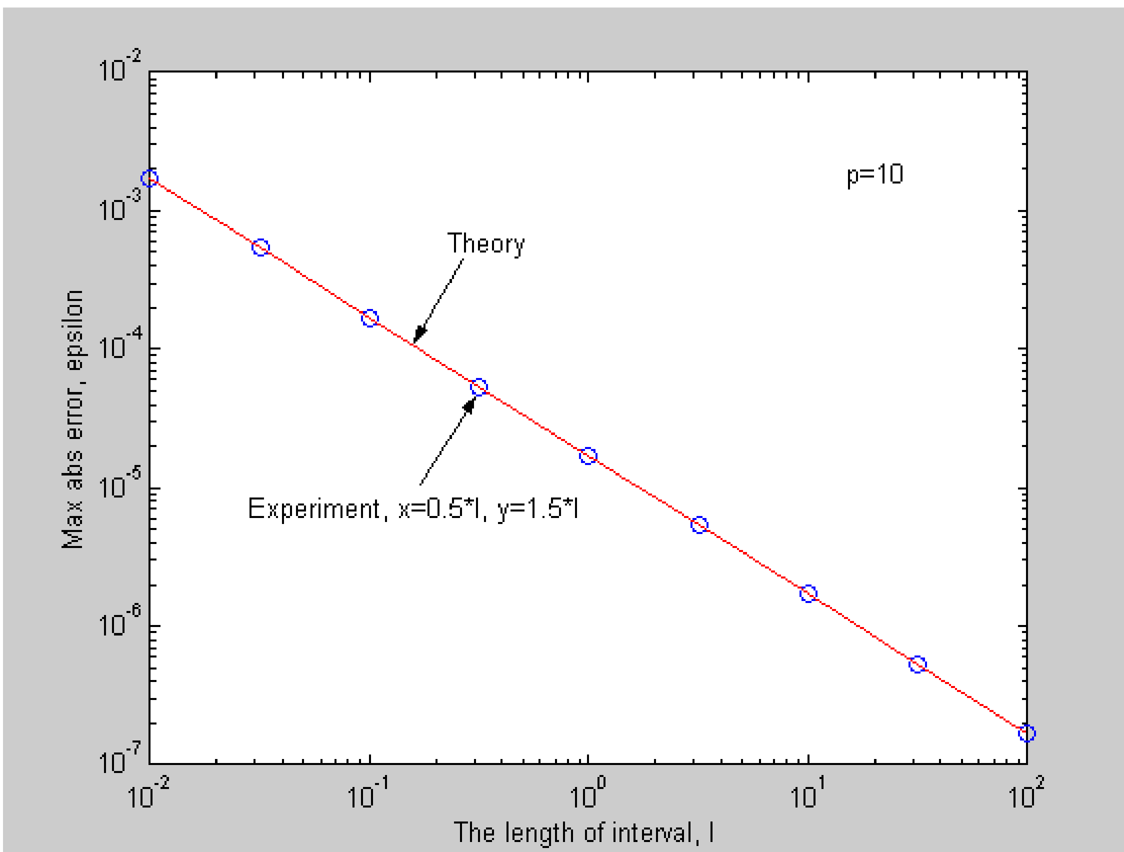

10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
Max abs error, epsilon
p=10
Theory
Experiment, x=0.5*l, y=1.5*l
10^-2
10^-1
10^0
10^1
10^2
The length of interval, l

Figure 2:

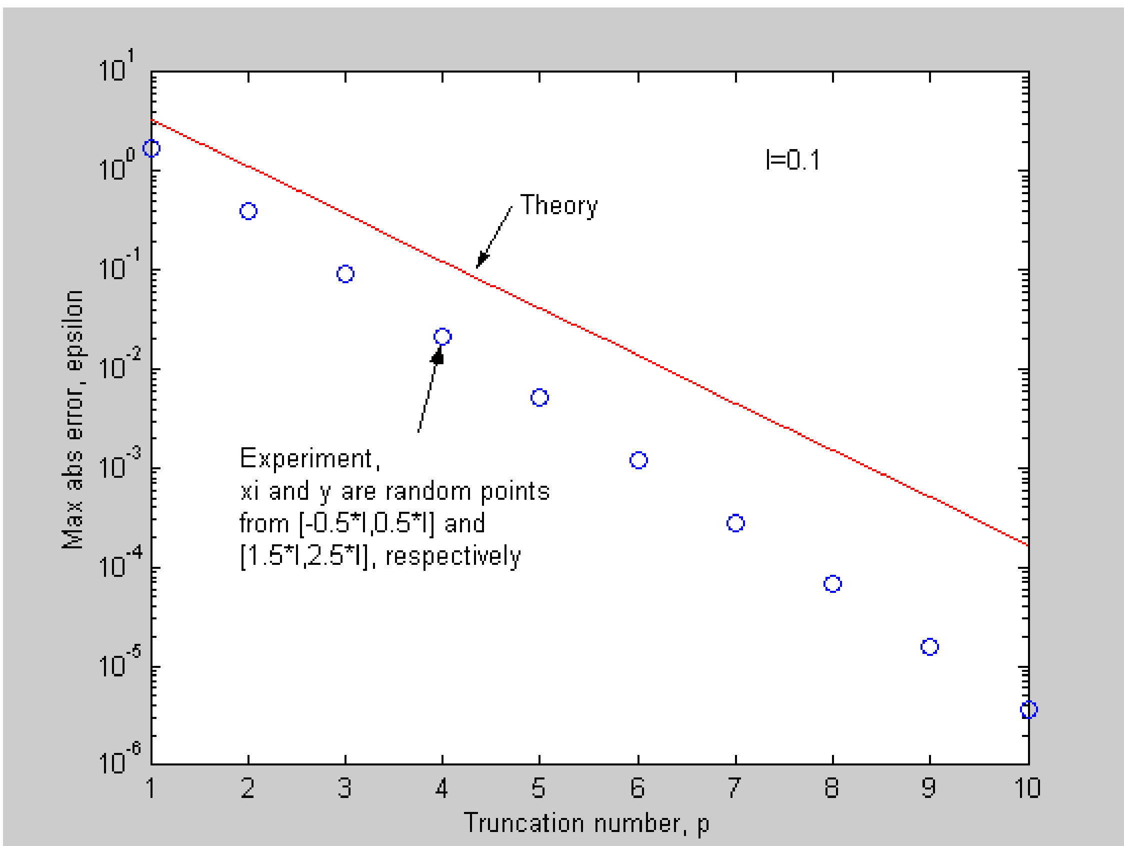

Max abs error, epsilon
Truncation number, p
l=0.1
Theory
Experiment,
xi and y are random points
from [-0.5*l,0.5*l] and
[1.5*l,2.5*l], respectively

Figure 3:

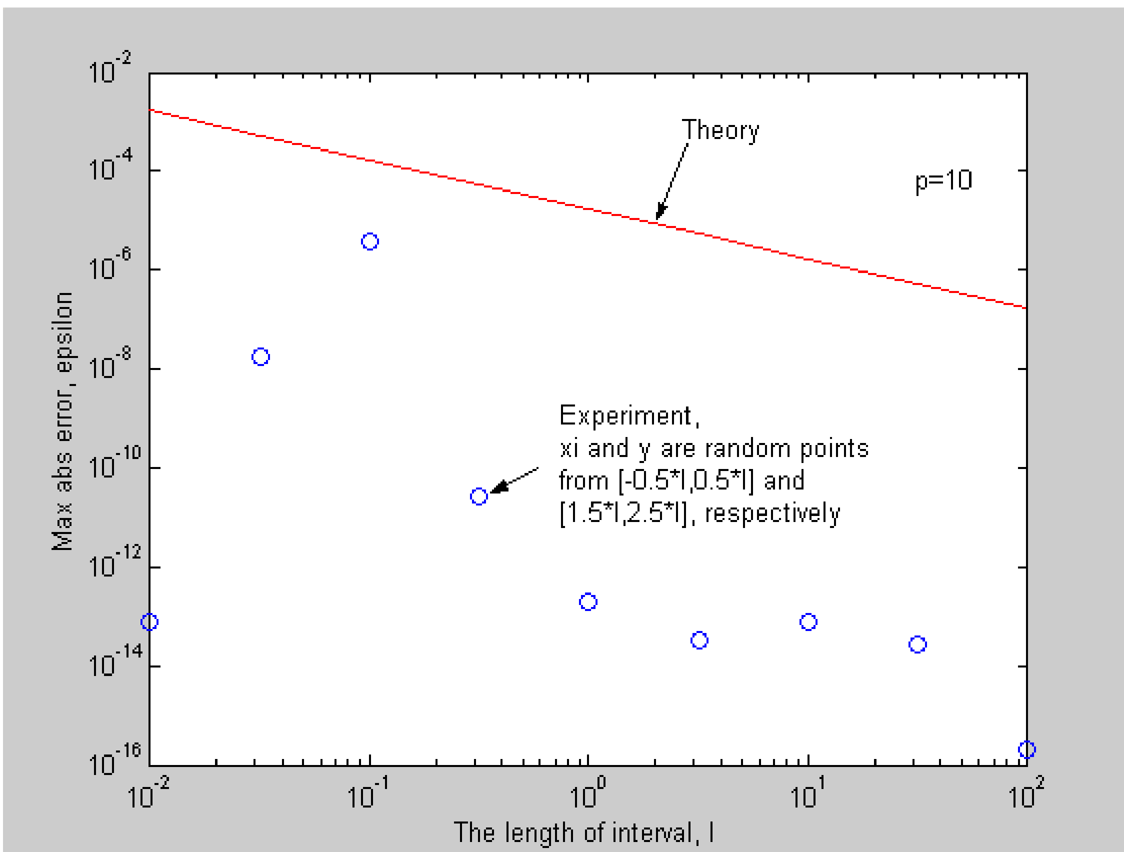

Theory
p=10
Experiment,
xi and y are random points
from [-0.5*l,0.5*l] and
[1.5*l,2.5*l], respectively
Max abs error, epsilon
The length of interval, l
10^-2
10^-4
10^-6
10^-8
10^-10
10^-12
10^-14
10^-16
10^-1
10^0
10^1
10^2

Figure 4:

2. The R expansion is for $(y - x_*) < (x_i - x_*)$, and is given by

$$\frac{1}{y - x_i} = \frac{1}{(y - x_*) - (x_i - x_*)} = \frac{1}{-(x_i - x_*)\left(1 - \frac{y - x_*}{x_i - x_*}\right)}$$

$$= \frac{-1}{(x_i - x_*)}\left(1 + \frac{y - x_*}{x_i - x_*} + \ldots + \left(\frac{y - x_*}{x_i - x_*}\right)^{p-1}\right) - \frac{1}{(x_i - x_*)}\frac{1}{1 - \frac{y - x_*}{x_i - x_*}}\left(\frac{y - x_*}{x_i - x_*}\right)^p$$

$$= \sum_{m=0}^{p-1} -\frac{(y - x_*)^m}{(x_i - x_*)^{m+1}} + \frac{1}{y - x_i}\left(\frac{y - x_*}{x_i - x_*}\right)^p$$

$$a_m(x, x_*) = -(x_i - x_*)^{-m-1} \quad R_m(y - x_*) = (y - x_*)^m \quad Error(p) = \frac{1}{y - x_i}\left(\frac{y - x_*}{x_i - x_*}\right)^p$$

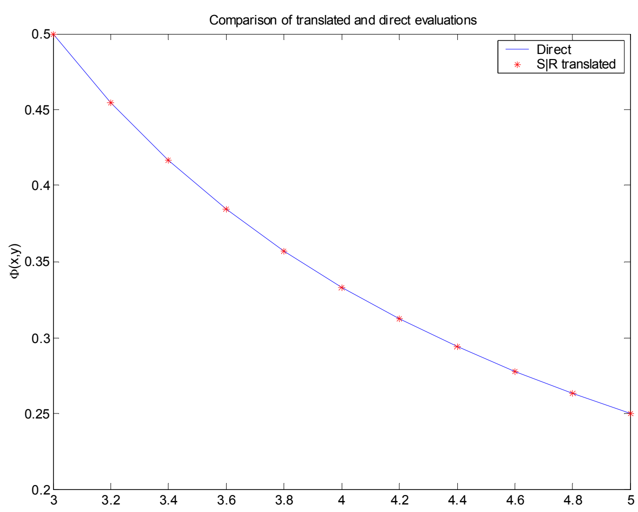
For doing the p-truncated $S|R$-translation of S-expansion coefficients to R-expansion coefficients we use the expression for the matrix given in class

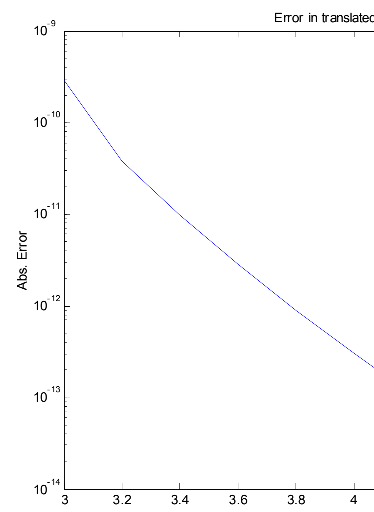
$$(S|R)_{mn}(t) = \frac{1}{m!}\frac{d^m S_n(t)}{dt^n} = \frac{(-1)^m (m+n)!}{m! n! t^{n+m+1}}$$

$$(\mathbf{S}|\mathbf{R})(t) = \begin{pmatrix} t^{-1} & t^{-2} & t^{-3} & \ldots \\ -t^{-2} & -2t^{-3} & -3t^{-4} & \ldots \\ t^{-3} & 3t^{-4} & 6t^{-5} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

A matlab function to calculate this matrix and return the R coefficients after a matrix vector product is attached.

3. For $l = 2$ we show the results of the $p = 20$ truncated S-expansion of $\Phi(y, x_i)$ shifted by the vector $t = 4$. Convolve the output vector of R-expansion coefficients with R-basis functions centered at $x_* = 2l$ to get approximate value of $\Phi(y, x_i)$ at $y \in \left[\frac{3}{2}l, \frac{5}{2}l\right]$. Compare this result with exact (straightforward) computation of $\Phi(y, x_i)$.





4. **Method 1. Error Bound Using Operator Norms.** Truncation of R-series introduces error. If

translation is performed exactly we have the following series:

$$\begin{aligned}\Phi\left(y,x_i\right) &= \sum_{m=0}^{\infty} a_m(x_i,x_*) R_m\left(y-x_*\right),\\ a_m(x_i,x_*) &= -\left(x_i-x_*\right)^{-m-1}, \quad m=0,1,...,\\ R_m\left(y-x_*\right) &= \left(y-x_*\right)^m, \quad m=0,1,...\end{aligned}$$

which should be considered near $x_*=2l$. Truncation error of the exact expansion is

$$\begin{aligned}\left|\Phi\left(y,x_i\right)-\sum_{m=0}^{p-1} a_m(x_i,x_*) R_m\left(y-x_*\right)\right| &= \left|\sum_{m=p}^{\infty} a_m(x_i,x_*) R_m\left(y-x_*\right)\right| = \left|-\frac{1}{x_i-x_*}\sum_{m=p}^{\infty}\left(\frac{y-x_*}{x_i-x_*}\right)^m\right|\\ &= \left|\frac{1}{y-x_i}\left(\frac{y-x_*}{x_i-x_*}\right)^p\right| = \left|\frac{1}{y-x_i}\right|\left|\left(\frac{y-x_*}{x_i-x_*}\right)^p\right| \leqslant \frac{1}{l}\frac{1}{3^p}.\end{aligned}$$

This shows that the error of the $p-$truncation operator both for $S-$ and $R-$ series is the same, so we have

$$\left|\Pr(p)\Phi\left(y,x_i\right)-\Phi\left(y,x_i\right)\right| \leqslant \epsilon, \quad \epsilon=\frac{1}{l}\frac{1}{3^p}.$$

So the norm of the truncation operation, $\Pr(p)$ does not exceed $1+\epsilon$. Also $\|\mathcal{T}(t)\|\leqslant 1$,so

$$\|\Pr(p)\|\,\|\mathcal{T}(t)\| \leqslant 1+\epsilon.$$

Consider now the error of the p-truncated translation:

$$\begin{aligned}\left|\mathcal{T}^{(p)}(t)\Phi\left(y,x_i\right)-\mathcal{T}(t)\Phi\left(y,x_i\right)\right| &= \left|\Pr(p)\mathcal{T}(t)\Pr(p)\Phi\left(y,x_i\right)-\mathcal{T}(t)\Phi\left(y,x_i\right)\right| =\\ &= |\Pr(p)\mathcal{T}(t)\Pr(p)\Phi\left(y,x_i\right)-\Pr(p)\mathcal{T}(t)\Phi\left(y,x_i\right)+\Pr(p)\mathcal{T}(t)\Phi\left(y,x_i\right)-\mathcal{T}(t)\Phi\left(y,x_i\\ &\leqslant \left|\Pr(p)\mathcal{T}(t)\Pr(p)\Phi\left(y,x_i\right)-\Pr(p)\mathcal{T}(t)\Phi\left(y,x_i\right)\right|+|\Pr(p)\mathcal{T}(t)\Phi\left(y,x_i\right)-\mathcal{T}(t)\Phi\left(y,x\\ &\leqslant \left|\Pr(p)\mathcal{T}(t)\left(\Pr(p)\Phi\left(y,x_i\right)-\Phi\left(y,x_i\right)\right)\right|+\left|\Pr(p)\Phi\left(y,x_i\right)-\Phi\left(y,x_i\right)\right|\\ &\leqslant \|\Pr(p)\|\,\|\mathcal{T}(t)\|\left|\left(\Pr(p)\Phi\left(y,x_i\right)-\Phi\left(y,x_i\right)\right)\right|+\left|\Pr(p)\Phi\left(y,x_i\right)-\Phi\left(y,x_i\right)\right|\\ &\leqslant (1+\epsilon)\epsilon+\epsilon=2\epsilon+\epsilon^2\approx 2\epsilon.\end{aligned}$$

4. **Method 2. Tighter Error Bound. Direct Evaluation.** Consider S-expansion near point x_{*1} (in our case $x_{*1}=0$) :

$$\begin{aligned}\Phi\left(y,x_i\right) &= \sum_{m=0}^{\infty} b_m(x_i,x_{*1}) S_m\left(y-x_{*1}\right),\\ b_m(x_i,x_{*1}) &= \left(x_i-x_{*1}\right)^m, \quad m=0,1,...,\\ S_m\left(y-x_{*1}\right) &= \left(y-x_{*1}\right)^{-m-1}, \quad m=0,1,...\end{aligned}$$

Exact S|R-translation to new expansion center x_{*2} (in our case $x_{*2}=2l=t$) can be performed using infinite matrix

$$(S|R)_{mn}(t)=\frac{(-1)^m(m+n)!}{m!n!t^{m+n+1}}, \quad t=x_{*2}-x_{*1}$$

so

$$a_m(x_i,x_{*2})=\sum_{n=0}^{\infty}(S|R)_{mn}(t)\,b_n(x_i,x_{*1}).$$

In this case we have

$$\Phi\left(y,x_i\right)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}c_{mn},$$

where

$$\begin{aligned} c_{mn} &= (S|R)_{mn}(t)\,b_n(x_i,x_{*1})R_m\left(y-x_{*2}\right) \\ &= \frac{(-1)^m(m+n)!}{m!n!t^{m+n+1}}\left(x_i-x_{*1}\right)^n\left(y-x_{*2}\right)^m \end{aligned}$$

Translation with $p-$truncated operator $(S|R)^{(p)}_{mn}(t)$ yields

$$\Phi^{(p)}\left(y,x_i\right)=\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}c_{mn}.$$

The error of truncated translation is therefore

$$\begin{aligned}
\left|\Phi\left(y,x_i\right)-\Phi^{(p)}\left(y,x_i\right)\right| &= \left|\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}c_{mn}-\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}c_{mn}\right| \\
&= \left|\sum_{m=0}^{p-1}\sum_{n=0}^{\infty}c_{mn}+\sum_{m=p}^{\infty}\sum_{n=0}^{\infty}c_{mn}-\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}c_{mn}\right| \\
&= \left|\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}c_{mn}+\sum_{m=0}^{p-1}\sum_{n=p}^{\infty}c_{mn}+\sum_{m=p}^{\infty}\sum_{n=0}^{\infty}c_{mn}-\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}c_{mn}\right| \\
&= \left|\sum_{m=0}^{p-1}\sum_{n=p}^{\infty}c_{mn}+\sum_{m=p}^{\infty}\sum_{n=0}^{\infty}c_{mn}\right| \leqslant \sum_{m=0}^{p-1}\sum_{n=p}^{\infty}\left|c_{mn}\right|+\sum_{m=p}^{\infty}\sum_{n=0}^{\infty}\left|c_{mn}\right| \\
&\leqslant \sum_{m=0}^{\infty}\sum_{n=p}^{\infty}\left|c_{mn}\right|+\sum_{m=p}^{\infty}\sum_{n=0}^{\infty}\left|c_{mn}\right| = \sum_{n=p}^{\infty}\sum_{m=0}^{\infty}\left|c_{mn}\right|+\sum_{m=p}^{\infty}\sum_{n=0}^{\infty}\left|c_{mn}\right| \\
&= \sum_{m=p}^{\infty}\sum_{n=0}^{\infty}\left|c_{nm}\right|+\sum_{m=p}^{\infty}\sum_{n=0}^{\infty}\left|c_{mn}\right| = \sum_{m=p}^{\infty}\sum_{n=0}^{\infty}\left(\left|c_{nm}\right|+\left|c_{mn}\right|\right) \\
&= \sum_{m=p}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n)!}{m!n!t^{m+n+1}}\left[\left|x_i-x_{*1}\right|^n\left|y-x_{*2}\right|^m+\left|x_i-x_{*1}\right|^m\left|y-x_{*2}\right|^n\right] \\
&\leqslant \sum_{m=p}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n)!}{m!n!t^{m+n+1}}\left[\left|\frac{l}{2}\right|^n\left|\frac{l}{2}\right|^m+\left|\frac{l}{2}\right|^m\left|\frac{l}{2}\right|^n\right] \\
&= 2\sum_{m=p}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n)!l^{n+m}}{2^{n+m}m!n!t^{m+n+1}} = 2\sum_{m=p}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n)!l^{n+m}}{2^{n+m}m!n!\left(2l\right)^{m+n+1}} \\
&= \frac{1}{l}\sum_{m=p}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n)!}{m!n!}\left(\frac{1}{4}\right)^{n+m} = \frac{1}{l}\sum_{m=p}^{\infty}\left(\frac{1}{4}\right)^m\sum_{n=0}^{\infty}\frac{(m+n)!}{m!n!}\left(\frac{1}{4}\right)^n \\
&= \frac{1}{l}\sum_{m=p}^{\infty}\left(\frac{1}{4}\right)^m\frac{1}{\left(1-\frac{1}{4}\right)^{m+1}} = \frac{1}{l}\sum_{m=p}^{\infty}\left(\frac{1}{4}\right)^m\frac{1}{\left(\frac{3}{4}\right)^{m+1}} \\
&= \frac{4}{l}\sum_{m=p}^{\infty}\frac{1}{3^{m+1}} = \frac{4}{l}\frac{1}{3^{p+1}}\sum_{l=0}^{\infty}\frac{1}{3^l} = \frac{4}{l}\frac{1}{3^{p+1}}\frac{1}{1-\frac{1}{3}} = \frac{4}{l}\frac{1}{3^{p+1}}\frac{1}{\frac{2}{3}} \\
&= \frac{2}{l}\frac{1}{3^p} = 2\epsilon. \quad \left(\epsilon=\frac{1}{l}\frac{1}{3^p}\right).
\end{aligned}$$

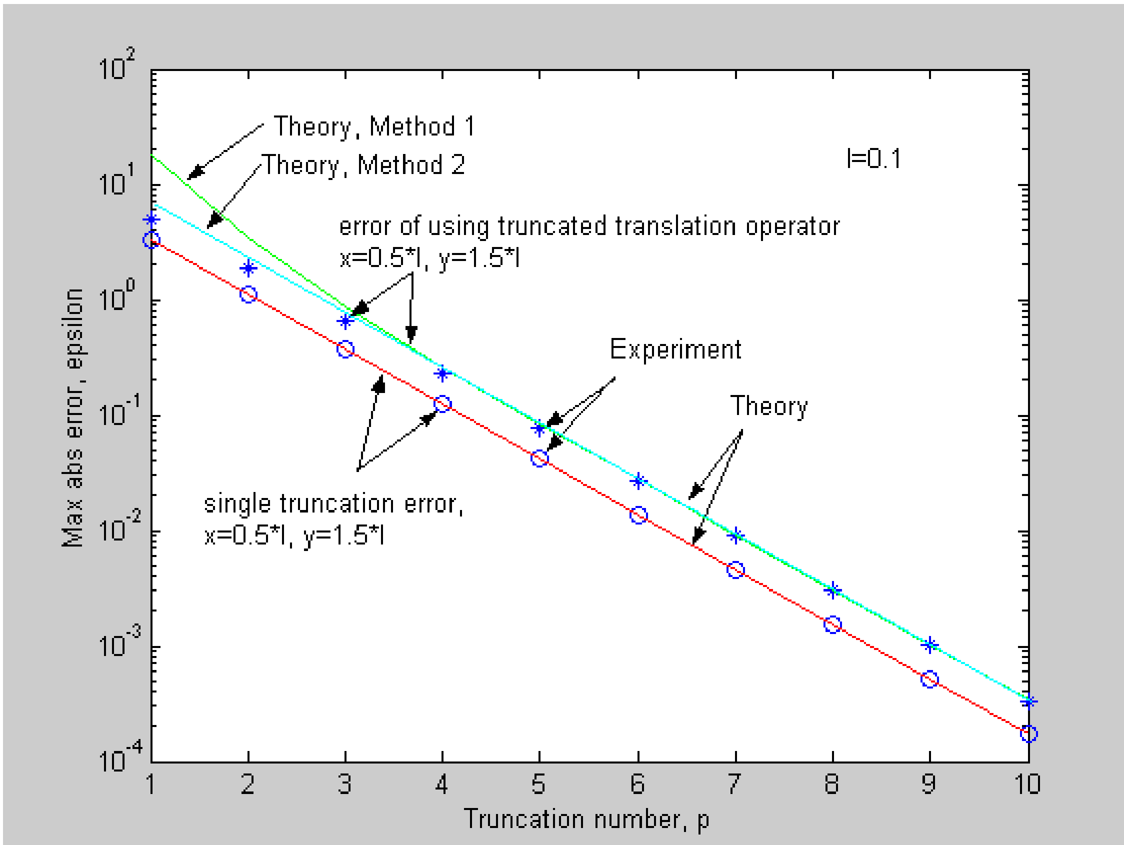


Figure 5:

Here we used the fact that

$$\frac{1}{(1-\alpha)^{m+1}} = 1 + (m+1)\,\alpha + \frac{(m+1)\,(m+2)}{2!}\alpha^2 + ... = \sum_{n=0}^{\infty} \frac{(m+n)!}{m!n!}\alpha^n, \quad |\alpha| < 1.$$

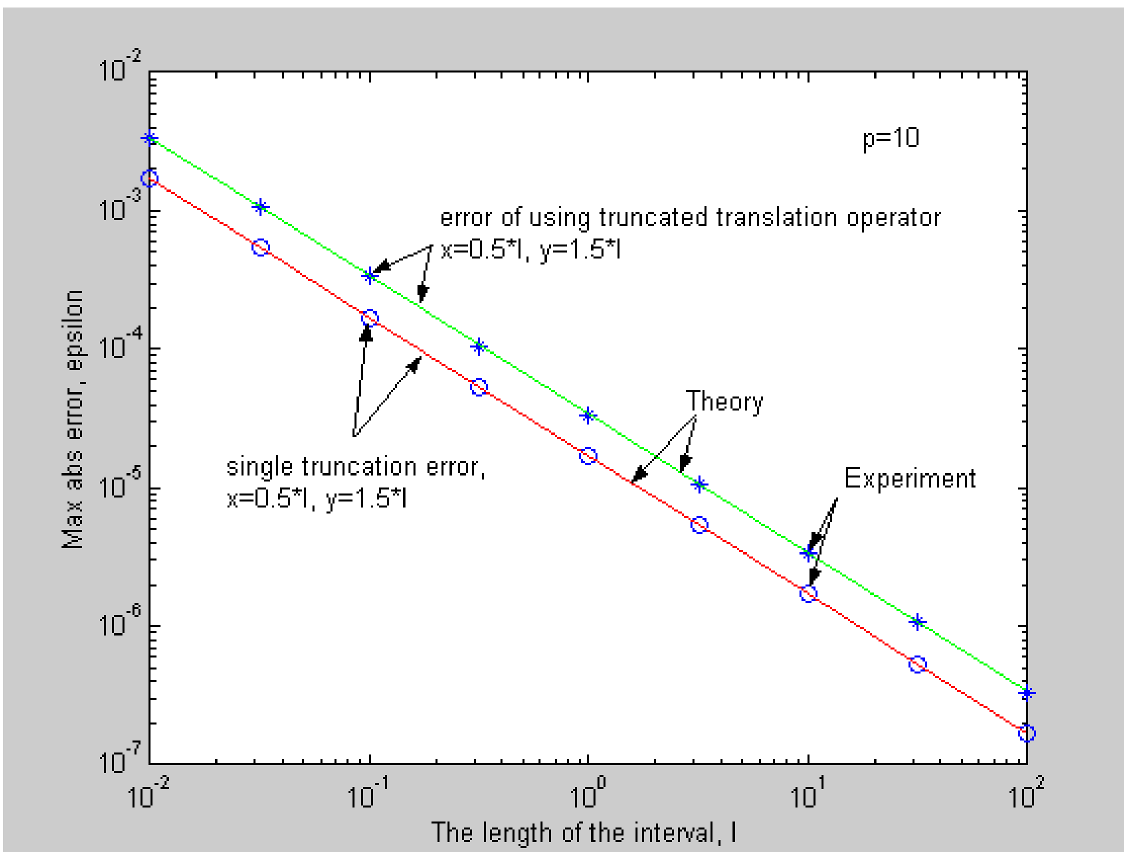

p=10
error of using truncated translation operator
x=0.5*l, y=1.5*l
single truncation error,
x=0.5*l, y=1.5*l
Theory
Experiment
Max abs error, epsilon
The length of the interval, l
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
10^-1
10^0
10^1
10^2

Figure 6:

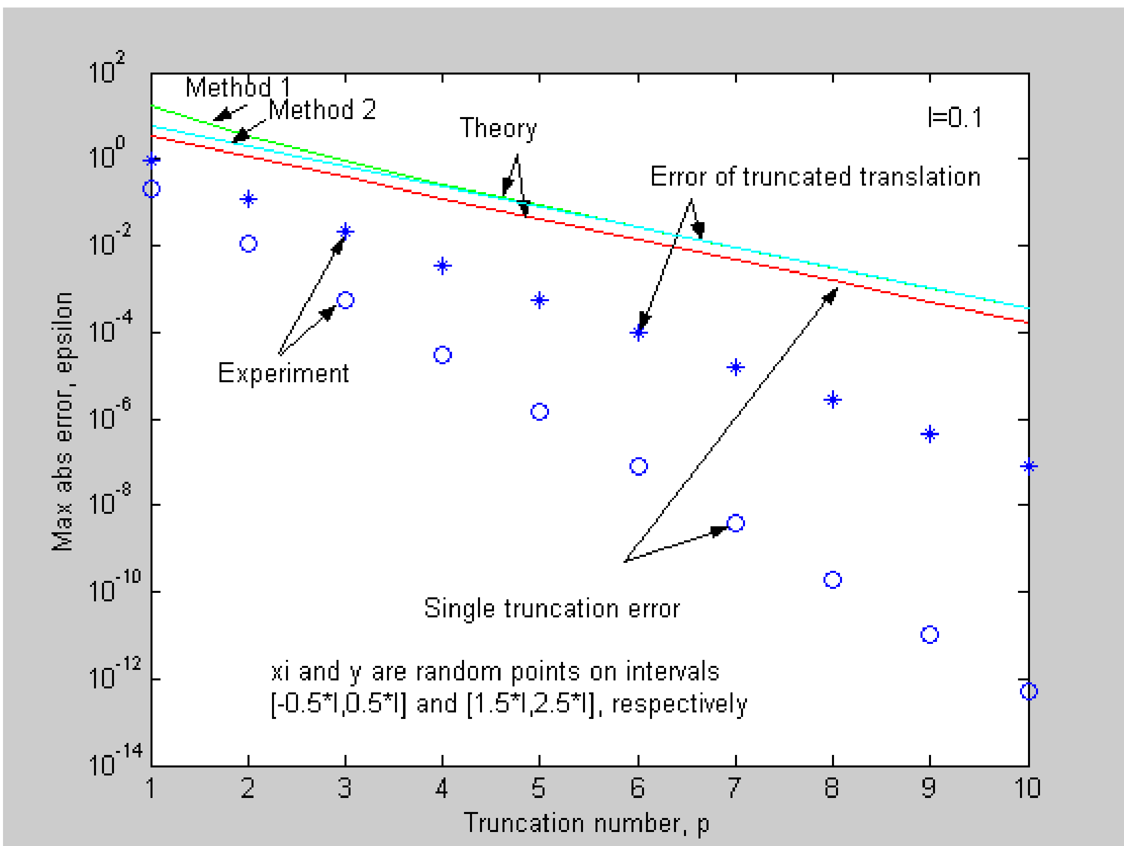

Method 1
Method 2
Theory
l=0.1
Error of truncated translation
Experiment
Single truncation error
xi and y are random points on intervals
[-0.5*l,0.5*l] and [1.5*l,2.5*l], respectively
Max abs error, epsilon
Truncation number, p
10^2
10^0
10^-2
10^-4
10^-6
10^-8
10^-10
10^-12
10^-14
1
2
3
4
5
6
7
8
9
10

Figure 7:

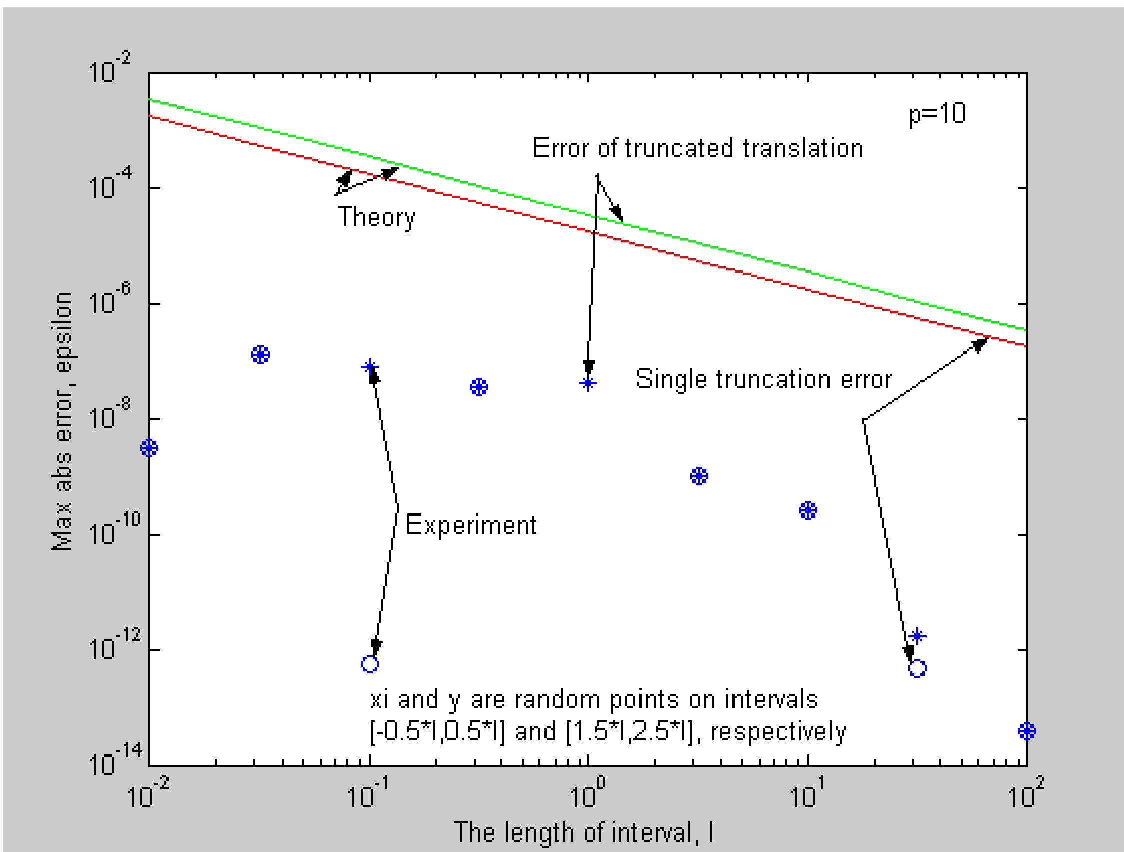

p=10
Error of truncated translation
Theory
Single truncation error
Experiment
xi and y are random points on intervals
[-0.5*l,0.5*l] and [1.5*l,2.5*l], respectively
Max abs error, epsilon
The length of interval, l
10^-2
10^-4
10^-6
10^-8
10^-10
10^-12
10^-14
10^-1
10^0
10^1
10^2

Figure 8: