Computational Methods
CMSC/AMSC/MAPL 460

Polynomial Interpolation

Ramani Duraiswami,
Dept. of Computer Science

Interpolation

- Given a function at *N* points, find its value at other point(s)
  - "within" the points
    - Interpolation, Imputation
  - "outside" the points
    - Extrapolation, Forecasting
- How do we extend the value from known points to unknown points?
  - Have to have prior knowledge about the function
  - Can do what is convenient
- Occam's razor
  - Parsimony: "One should not increase, beyond what is necessary, the number of entities required to explain anything"
  - *Key in all scientific modeling*

Function representation: in a "function space"

- In interpolation sampled values of a function are available
- We need to extend these values to points where they are not available
- Can assume that the function is expanded over a basis of functions which span the functional space
  - Polynomials are a basis -- Fourier series are another
  - Many others
- Knowing coefficients specifies the function
- Most common basis are power series/polynomials
  - Here x_* is a point about which we expand in series
  - a_m are coefficients

$$f(y - x_*) = \sum_{m=0}^{\infty} a_m (y - x_*)^m,$$

- Polynomial interpolation is often natural
  - Part of theory of Taylor series, solution of differential equations via power series, in computing integrals
  - A theorem guaranteeing that a polynomial can represent a function is available

Interpolation

- Weierstrass theorem

Theorem: (Weierstrass) For all $f \in \mathcal{C}[a, b]$, for all $\epsilon > 0$, there exists a degree n and a polynomial p_n such that $||f - p||_\infty < \epsilon$.

- Provides a guarantee that polynomials can interpolate any function
- On the other hand does not tell us how to choose the polynomial
- Also does not guarantee that the polynomial will actually "interpolate" the function … only that it will be within ε.
- Does not tell what the degree of the polynomial is

Taylor Series

- Let $f(y)$ be a real function, $f(y) \in C^n\,[x_*, x_*+r)$
  - n^{th} derivative f^n exists for $x_* \le y < x_*+r$

$$f(y) = f(x_*) + f'(x_*)(y - x_*) + \frac{1}{2!}f''(x_*)(y - x_*)^2 + \ldots + \frac{1}{(n-1)!}f^{(n-1)}(x_*)(y - x_*)^{n-1} + \text{Residual}_n(y).$$

  - Residual determines accuracy
  - Two evaluations of remainder
    - Cauchy evaluation
    - Lagrange evaluation

$$|\text{Residual}_n(y)| \leqslant \frac{|y - x_*|^n}{n!} \sup_{x_* < y < x_* + r_*} |f^{(n)}(y)|.$$

$$\text{Residual}_n(y) = \int_{x_*}^{y} dx \int_{x_*}^{x} dx \ldots \int_{x_*}^{x} f^{(n)}(x)dx = \frac{1}{n!}f^{(n)}(X)(y - x_*)^n,$$

$$X \in (x_*, x_* + r_*).$$

Polynomial Facts

- A polynomial of degree k has at most k distinct zeroes, unless it is identically zero
- Sum of two polynomials of degree k is another polynomial of degree at most k.
- Polynomials can be expressed in many ways

For example $(x-2)(x-5) = x^2 - 2x - 5(x-2) = x^2 - 7x + 10$.

We have used three different sets of **basis functions** in this example:

1. $(x-2)(x-5)$, $x-1$, and 1.
2. x^2, x, and $x-2$.
3. x^2, x, and 1.
   - Degree of basis functions is 2, 1 and 0 …
   - Basis 3 is the power basis or monomial basis
   - Any basis can be used … often "orthogonal polynomials" are used

Uniqueness of interpolant

- We know that the polynomial exists
- Suppose that there are two different polynomials that can interpolate the data
- Let them be p_{n-1} and q_{n-1} .
- So we have $p_{n-1}(x_i) = y_i,\ i=1,\ \ldots n$

  $$q_{n-1}(x_i) = y_i,\ i=1,\ \ldots n$$
- So $p_{n-1}(x_i) - q_{n-1}(x_i) = 0,\ i=1,\ \ldots n$
- $p_{n-1} - q_{n-1}$ is the difference of two polynomials of degree $n-1$.
- It has n zeroes.
- Recall polynomial of degree k has at most k zeroes, or is the zero polynomial.
- Here we have more zeroes than degree … so it is the zero polynomial
- So interpolant is unique

Interpolating polynomials in power form

- Given n values of the function y_i at points x_i
- Fit a polynomial $P(x)$ that interpolates data at these points
- If we have n points to interpolate, then a polynomial of degree n-1 is

$$P(x) = c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_{n-1} x + c_n$$

- We can write the condition that it interpolate as a linear system

$$\begin{pmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & x_2 & 1 \\ \cdots & \cdots & \cdots & \cdots & 1 \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$