Homework #9: 3.7.11, 3.8.4, 4.1.5, and 4.2.7 (due Thursday March 12).

3.7.11.

\[ U. \] **uniform dist \[ [-1, 1] \].

\[ W = g(U) = \begin{cases} 0 & U < 0 \\ U & U \geq 0 \end{cases} \]

\[ F_W(w), \ E[W] = ? \]

\[ F_W(w) = P[W \leq w] \]

\[ \Rightarrow P[W = 0] = P[-1 \leq U \leq 0] = \int_{-1}^{0} \frac{1}{2} \, du = \frac{1}{2} \]

\[ \Rightarrow P[W \leq w] = P[W = 0] + P[0 < W \leq w] \]

\[ = P[W = 0] + P[0 < U < M] \]

\[ \Rightarrow \int_{0}^{w} \frac{1}{2} \, du = \frac{w}{2} \]

\[ F_W(w) = \begin{cases} 0 & w < 0 \\ \frac{w}{2} & 0 \leq w \leq 1 \\ 1 & w > 1 \end{cases} \]

\[ E[W] = 0 \times P[W = 0] + \int_{0}^{1} w \cdot f_W(w) \, dw \]

\[ = 0 + \frac{1}{4} \left[ w^2 \right]_0^1 = \frac{1}{4} \]
3.8.4. \( W \sim \mathcal{N}(0, \sigma^2 = 6) \) Gaussian R.V.,

\[ C = \{ W > 0 \} . \]

(a). \( \frac{f_{W|C}(w)}{f_W(w)} \)

\[ \Rightarrow P \left[ W = w' \mid W > 0 \right] = \chi, \]

\[ \Rightarrow w' > 0 \Rightarrow \chi > 0. \]

\[ \begin{align*}
\[ w' < 0 \Rightarrow \chi = 0 \]
\end{align*} \]

\[ \Rightarrow \frac{P \left[ W = w' \right]}{P \left[ W > 0 \right]} = \left\{ \begin{array}{ll}
\frac{1}{\sqrt{2\pi}} e^{-\frac{w'^2}{2}} & w' > 0, \\
0 & w' < 0,
\end{array} \right. \]

(b). \( E[W \mid C] = \int_0^\infty w' \chi \frac{1}{\sqrt{2\pi}} e^{-\frac{w'^2}{2}} dw' \)

\[ = 2 \int_0^\infty w' \frac{1}{\sqrt{2\pi}} e^{-\frac{w'^2}{2}} dw' \]

\[ \begin{align*}
\frac{w'^2}{2} & = v \\
\frac{1}{16} w dw = dv, \\
w dw & = 16 dv
\end{align*} \]

\[ = 2 \int_0^\infty \frac{1}{\sqrt{2\pi}} 16 e^{-v} dv. \]

\[ = \frac{\sqrt{32}}{\pi} \int_0^\infty e^{-v} dv = \frac{\sqrt{32}}{\pi}. \]
(c) \( \text{Var}[W|c] \)

\[= \int_{-\infty}^{\infty} w^2 f_w(w) \, dw - \left( \int_{-\infty}^{\infty} w f_w(w) \, dw \right)^2 \]

\[= \text{Var}[W] \]

\[= 1/6. \]

\[\therefore \text{Var}[W|c] = \mathbb{E}[w^2|c] - \left\{ \mathbb{E}[W|c] \right\}^2 \]

\[= 1/6 - \frac{3^2}{16} = 5/8. \]
4.1.5.

(a) \( A = \{ x \leq x_1, y \leq y_1 \leq y_2 \} \), \( B = \{ x_1 \leq x \leq x_2, y \leq y_1 \} \), \( C = \{ x_1 \leq x \leq x_2, y_1 \leq y \leq y_2 \} \).

(b) \( P[A] = F_{x,y}(x_1, y_2) - F_{x,y}(x_1, y_1) \).

\( P[B] = F_{x,y}(x_2, y_1) - F_{x,y}(x_1, y_1) \).

\( P[C] = F_{x,y}(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F_{x,y}(x_1, y_1) \).

\( P[A \cup B \cup C] = F_{x,y}(x_2, y_2) - F_{x,y}(x_1, y_1) \).

(c) Since \( A, B, \) and \( C \) are mutually exclusive,

\[ P[A \cup B \cup C] = P[A] + P[B] + P[C] \]

However, since we want to express

\[ P[C] = P[x_1 < X < x_2, y_1 < Y \leq y_2] \]

in terms of the joint CDF \( F_{X,Y}(x, y) \), we write

\[ P[C] - P[A \cup B \cup C] = P[A] - P[B] \]

\[ = F_{X,Y}(x_2, y_2) - F_{X,Y}(x_1, y_2) - F_{X,Y}(x_2, y_1) + F_{X,Y}(x_1, y_1) \]

which completes the proof of the theorem.
$4.2. \pi.$

\[
\begin{align*}
P_{k,x}(k,x) & \quad \text{for } k \text{-rejected circuits among } n-1 \text{ circuits rejected.} \\
P_{k,x}(k,0) & \quad \Theta \Theta \ldots \Theta \quad \text{for } k \text{-rejected circuits among } n-1 \text{ circuits accepted.} \\
\Rightarrow & \quad \left\{ \\
P_{k,x}(k,0) & = \binom{n-1}{k-1} (1-p)^{k-1} p^{n-k}, \quad 0 \leq k \leq n, x = 0 \\
P_{k,x}(k,1) & = \binom{n-1}{k} (1-p)^{k} p^{n-k-1}, \quad 0 \leq k \leq n, x = 1 \\
\right. \\
\Rightarrow & \quad \left\{ \\
P_{k,x}(k,x) & = \begin{cases} \\
\binom{n-1}{k-1} (1-p)^{k-1} p^{n-k} & 1 \leq k \leq n, x = 0 \\
\binom{n-1}{k} (1-p)^{k} p^{n-k-1} & 0 \leq k \leq n, x = 1 \\
0 & \text{otherwise.} 
\end{cases}
\end{align*}
\]

\[k\] the number of circuits rejected,

\[x\] the number of acceptable circuit in the last first.